

HOMOTOPY GROUPS, FOCAL POINTS AND TOTALLY GEODESIC IMMERSIONS

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ABSTRACT. Let $f : \Sigma^{m-1} \rightarrow M^m$ be a totally geodesic immersion of a closed manifold Σ in a complete Riemannian manifold M and $g : N^n \rightarrow M^m$ an isometric immersion without focal points of a complete manifold N . If Σ has finite fundamental group then N is compact with finite fundamental group and we have:

- (1) if $m - n = 1$ then Σ and N have the same universal covering, and the homomorphism $f_*^i : \pi_i(\Sigma) \rightarrow \pi_i(M)$ is a monomorphism for $i = 1$ and an isomorphism for $i \geq 2$;
- (2) if $m - n \geq 2$ then it holds that:
 - (a) $f(\Sigma) \cap g(N) = \emptyset$;
 - (b) M is noncompact with finite fundamental group;
 - (c) the homomorphism $\iota_*^i : \pi_i(f(\Sigma)) \rightarrow \pi_i(M)$, induced by the inclusion $\iota : f(\Sigma) \rightarrow M$, is surjective for $1 \leq i \leq m - n - 1$;
 - (d) if $m - n = 2$ then $f_*^i : \pi_i(\Sigma) \rightarrow \pi_i(M)$ is a monomorphism for $i = 2$ and an isomorphism for $i \geq 3$;
- (3) if $m - n \geq 3$ then the following statements hold:
 - (a) the map $f_*^i : \pi_i(\Sigma) \rightarrow \pi_i(M)$ is a monomorphism for $i = 1$, an isomorphism for $2 \leq i \leq m - n - 2$ and an epimorphism for $i = m - n - 1$;
 - (b) if f is an embedding then $f_*^1 : \pi_1(\Sigma) \rightarrow \pi_1(M)$ is surjective and this, together with Item (a) above, implies that f is $(m - n - 1)$ -connected.

1. Introduction

Unless otherwise stated, all manifolds in this paper will be assumed to be connected. After a generalization of a result of Hadamard about the intersection of geodesics in a convex surface, Frankel [Fr] proved the following theorem.

Theorem A. (Frankel [Fr]) *Let M^m be an m -dimensional closed Riemannian manifold of positive curvature, Σ a closed manifold and $f : \Sigma \rightarrow M$ a totally geodesic immersion with codimension at most $m/2$. Then the induced homomorphism $f_*^1 : \pi_1(\Sigma) \rightarrow \pi_1(M)$ is surjective.*

There have been many generalizations of this theorem. Recently the relationship between the topology of totally geodesic submanifolds and the topology of ambient manifolds of positive curvature has been deepened in [Wi] and [FMR].

The assumption of positive curvature is essential in the above results. In fact, given any manifold M and any embedded closed submanifold Σ , one can construct metrics on M such that Σ is totally geodesic. Thus the only existence of a totally geodesic hypersurface does not imply any topological restriction. Our idea here is

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to replace the curvature hypothesis by the additional assumption of the existence of a complete isometric immersion $g : N \rightarrow M$ without focal points. Some examples in this introduction will illustrate this situation.

The work of Hermann ([He]) and its generalization by Bolton ([Bo]) show that the only existence of a complete isometric immersion $g : N \rightarrow M$ without focal points strongly relates the topologies of M and N (see Theorem B and Corollary 1.1 below). Our expectation when we began this work was that the union of the two hypotheses (existence of a totally geodesic hypersurface and a submanifold without focal points) should also restrict the relation between the topologies of M and Σ . In fact we obtain strong topological restrictions and even the compactness of N .

Let M and N be Riemannian manifolds and $g : N \rightarrow M$ an immersion. We will denote by \mathcal{N}_g the normal bundle of g . We recall the result of Bolton, which extends the work of Hermann ([He]):

Theorem B. (Bolton [Bo]) *Let M and N be complete Riemannian manifolds and $g : N \rightarrow M$ an isometric immersion without focal points. Then the normal exponential map $\exp^\perp : \mathcal{N}_g \rightarrow M$ is a covering map.*

Remark 1.1. As a direct consequence of Theorem B it follows that if M is simply-connected then $\exp^\perp : \mathcal{N}_g \rightarrow M$ is a diffeomorphism, which implies that: (a) g is an embedding, since it is the restriction of \exp^\perp to the submanifold $N \times \{0\} \subset \mathcal{N}_g$; (b) N is simply-connected, since it has the same homotopy type of \mathcal{N}_g .

For the seek of comparison with our results, we will enunciate the following consequence of Bolton's Theorem together with the fact that $g = \exp^\perp \circ j$, where $j : N \rightarrow \mathcal{N}_g$ is given by $j(x) = (x, 0)$.

Corollary 1.1. *Under the hypothesis of Bolton's Theorem the homomorphism $g_*^i : \pi_i(N) \rightarrow \pi_i(M)$ is an isomorphism for $i \geq 2$ and a monomorphism for $i = 1$.*

Now we recall some topological definitions. Consider path-connected topological sets A, B, C with $B \subset C$ and continuous maps $\alpha : A \rightarrow B$, $\iota : B \rightarrow C$ and $\beta : A \rightarrow C$ where ι is the inclusion map and $\beta = \iota \circ \alpha$. It is said that α is k -connected if $\alpha_*^i : \pi_i(A) \rightarrow \pi_i(B)$ is an isomorphism for $1 \leq i \leq k-1$ and an epimorphism for $i = k$. We say that the pair (C, B) is k -connected if the inclusion ι is k -connected, and it is well known that this is equivalent to the fact that the homotopy groups $\pi_i(C, B) = 0$ for $1 \leq i \leq k$. If α is a homeomorphism both notions are clearly equivalent. However, without this hypothesis, the k -connectedness of β is not equivalent to the k -connectedness of the pair $(C, \alpha(A))$, even if $\alpha(A) = B$. For example, consider the inclusion of a 2-torus $\iota : T \rightarrow \mathbb{R}^3$, and consider the universal covering $\alpha : \mathbb{R}^2 \rightarrow T$. Another example is to consider the universal covering $\alpha : S^n \rightarrow \mathbb{R}P^n$ of the n -dimensional real projective space and the identity map $\iota : \mathbb{R}P^n \rightarrow \mathbb{R}P^n$. We say that C is k -connected if the pair (C, p) is k -connected for some $p \in C$.

Our first result is the following.

Theorem 1.1. *Let $f : \Sigma^{m-1} \rightarrow M^m$ be a totally geodesic immersion of a closed manifold Σ in a complete Riemannian manifold M and $g : N^n \rightarrow M^m$ be an isometric immersion without focal points of a complete manifold N . If Σ has finite fundamental group then N is compact with finite fundamental group and we have:*

- (1) if $m - n = 1$ then Σ and N have the same universal covering, and the homomorphism $f_*^i : \pi_i(\Sigma) \rightarrow \pi_i(M)$ is a monomorphism for $i = 1$ and an isomorphism for $i \geq 2$;
- (2) if $m - n \geq 2$ then it holds that:
 - (a) $f(\Sigma) \cap g(N) = \emptyset$;
 - (b) M is noncompact with finite fundamental group;
 - (c) the homomorphism $\iota_*^i : \pi_i(f(\Sigma)) \rightarrow \pi_i(M)$, induced by the inclusion $\iota : f(\Sigma) \rightarrow M$, is surjective for $1 \leq i \leq m - n - 1$;
 - (d) if $m - n = 2$ then $f_*^i : \pi_i(\Sigma) \rightarrow \pi_i(M)$ is a monomorphism for $i = 2$ and an isomorphism for $i \geq 3$;
- (3) if $m - n \geq 3$ then the following statements hold:
 - (a) the map $f_*^i : \pi_i(\Sigma) \rightarrow \pi_i(M)$ is a monomorphism for $i = 1$, an isomorphism for $2 \leq i \leq m - n - 2$ and an epimorphism for $i = m - n - 1$;
 - (b) if f is an embedding then $f_*^1 : \pi_1(\Sigma) \rightarrow \pi_1(M)$ is surjective and this, together with Item (a) above, implies that f is $(m - n - 1)$ -connected.

Remark 1.2. In several points Theorem 1.1 may not be improved:

- (i) Example 1.2 below shows that the finiteness of $\pi_1(\Sigma)$ is an essential assumption;
- (ii) for the case $m - n = 1$, Example 1.1 below shows that $f(\Sigma)$ may intersect $g(N)$ and M may be compact with infinite fundamental group (compare with Item (2)); moreover, f_*^1 may be non-surjective even when f is an embedding (compare with Item (3)-(b));
- (iii) for the case $m - n = 2$, Example 1.3 will show that the map f_*^1 could be non-injective or non-surjective, and that f_*^2 could be non-surjective;
- (iv) for the case $m - n \geq 3$, Example 1.4 will show that f_*^{m-n-1} may be non-injective (compare with Item (3)-(a)); Example 1.5 will show that f_*^1 may be non-surjective, and it also shows that the embeddedness of f is essential in Item (3)-(b).

We think it is interesting to consider the following question.

Question 1.1. Assume the hypotheses of Theorem 1.1 with $m - n \geq 2$. Is that true that $g : N \rightarrow M$ is a homotopy equivalence?

By Corollary 1.1, this question is equivalent to ask if $g_*^1 : \pi_1(N) \rightarrow \pi_1(M)$ is an epimorphism. We notice that in the proof of Theorem 1.1 in [MM] this question was answered positively in the very particular case that N is a point. Example 1.1 below shows that the answer would be negative if we take $m - n = 1$. We recall that a topological space X is called aspherical if $\pi_i(X) = 0$ for $i \geq 2$. It is well known that if X is an aspherical manifold then its universal covering is homeomorphic to \mathbb{R}^n with $n \geq 0$. Thus the following corollary is a consequence of Theorem 1.1.

Corollary 1.2. The answer to Question 1.1 is positive if N is aspherical or if $f(\Sigma)$ is simply-connected.

The next examples were cited in Remark 1.2.

Example 1.1. Consider the Riemannian product $M = N' \times S^1$ where N' is a closed manifold with finite fundamental group and S^1 is a round circle. Note that $\Sigma = N' \times \{q\}$ has finite fundamental group. Take $N = \Sigma$ and let $f = g : N \rightarrow M$ be the inclusion map. The embedding g is free of focal points and f is totally

geodesic. Thus Theorem 1.1 applies for $m - n = 1$. Note that f_*^1 is not surjective, $f(\Sigma) \cap g(N) \neq \emptyset$ and M is compact with infinite fundamental group.

Example 1.2. Consider $M = N' \times T^k$ where N' is a closed manifold with infinite fundamental group and T^k is a k -dimensional flat torus with $k \geq 2$. Let $f : \Sigma = N' \times T^{k-1} \times \{q\} \rightarrow M$ be the inclusion map. Notice that f is a totally geodesic embedding. Choose $p \in T^{k-1}$. Let $g : N = N' \times \{(p, q)\} \rightarrow M$ be the inclusion map. The embedding g is free of focal points with codimension $k \geq 2$. Note that $\pi_1(\Sigma)$ is infinite, hence Theorem 1.1 does not apply. In fact, several conclusions in this theorem fail: N has infinite fundamental group; $f(\Sigma) \cap g(N) \neq \emptyset$; M is compact with infinite fundamental group; f is an embedding and f_*^1 is not surjective (compare with Items (2)-(c) and (3)-(b) of Theorem 1.1).

Example 1.3. Let $p : M^m \rightarrow N^n$ be a vector bundle over a manifold N and fix a smooth fiber metric $x \in N \mapsto \langle \cdot, \cdot \rangle_x$ where $\langle \cdot, \cdot \rangle_x$ is an inner product on the fiber V_x . Let $S_x \subset V_x$ be the unit sphere centered at the origin and set $\mathcal{S}_1 = \cup_{x \in N} S_x$. Let $g : N \rightarrow g(N) = N_0 \subset M$ be the null section. Proposition 4.1 will show that there exists a Riemannian metric ω on M such that \mathcal{S}_1 is a totally geodesic hypersurface of M and g is free of focal points. It holds also that $\exp^\perp : \mathcal{N}_g \rightarrow M$ is a diffeomorphism. If further N is compact then M is complete. This example may be used to see that some conclusions in Theorem 1.1 may not be improved (see Remark 1.2-(iii)). In fact, we first consider the particular case that M is the tangent bundle $M = TS^2$ equipped with the metric ω and $\Sigma = \mathcal{S}_1 = T_1 S^2 = \{(p, v) \in TS^2 \mid |v| = 1\}$. Let $f : \Sigma \rightarrow M$ be the totally geodesic inclusion map and $g : N \rightarrow N_0 = N \times \{0\} \subset TS^2$ the free of focal points null section. It is well known that Σ is diffeomorphic to SO_3 . We have that $\pi_1(\Sigma) = \pi_1(SO_3) = \mathbb{Z}_2$ and $\pi_1(M) = \pi_1(TS^2) = 0$. Furthermore, it holds that $\pi_2(\Sigma) = \pi_2(SO_3) = 0$ and $\pi_2(TS^2) = \mathbb{Z}$. Thus we conclude that f_*^1 is not injective and f_*^2 is not surjective. Now we consider the case that $(M, \omega) = (T(\mathbb{R}P^2), \omega)$ and $\mathcal{S}_1 = T_1(\mathbb{R}P^2) = \{(p, v) \in T(\mathbb{R}P^2) \mid |v| = 1\}$. Let $f : \Sigma = SU_2 \cong S^3 \rightarrow \mathcal{S}_1 \subset M$ be the universal covering with the induced metric and $g : \mathbb{R}P^2 \rightarrow \mathbb{R}P^2 \times \{0\} \subset M$ the natural embedding. Since $\pi_1(\Sigma) = 0$ and $\pi_1(M) = \pi_1(T(\mathbb{R}P^2)) = \mathbb{Z}_2$, the map f_*^1 is not surjective.

The following example is a typical situation where Theorem 1.1 holds and it will be used in Example 1.5.

Example 1.4. Define $M = \mathbb{R}^m$, with $m \geq 2$, endowed with the metric $ds^2 = dr^2 + \sigma^2(r)d\theta^2$ where $d\theta^2$ is the standard metric on the unit sphere S^{m-1} and $\sigma : [0, \infty) \rightarrow [0, \infty)$ is a smooth function satisfying $\sigma(r) > 0$ for all $r > 0$, $\sigma(0) = 0$, $\sigma'(0) = 1$ and $\sigma'(1) = 0$. We know that M is complete, the origin O is a pole (in particular, $N = \{O\}$ is free of focal points) and the sphere $\Sigma = S^{m-1}$ is totally geodesic. Notice that $\pi_{m-1}(\Sigma) = \mathbb{Z}$ and $\pi_{m-1}(M) = 0$. Thus f_*^{m-1} is not injective (see Remark 1.2-(iv)).

Example 1.5. Take $B = (\mathbb{R}^k, ds^2)$, with $k \geq 3$, and $ds^2 = dr^2 + \sigma^2(r)d\theta^2$ being the metric introduced in Example 1.4. Let $\mathcal{S} = S^{k-1} \subset B$ be the totally geodesic unit sphere and N' any Riemannian manifold. Consider a warped product $M = B \times_\rho N'$ and assume that the gradient $(\nabla \rho)|_{\mathcal{S}}$ is tangent to \mathcal{S} (for example, take $\rho(x) = \sigma(|x|^2)$). The manifold M is complete if N' is complete (see [O'N]). Let $g : N \rightarrow \{0\} \times N'$ and $f : \Sigma \rightarrow \mathcal{S} \times N'$ be any covering maps. Proposition 4.2 below will show that $g : N \rightarrow M$ is free of focal points and $f : \Sigma \rightarrow M$ is totally geodesic.

If further N is compact with finite fundamental group, then Theorem 1.1 applies. Now take $N = N' = \mathbb{R}P^n$ and $\Sigma = \mathcal{S} \times S^n$ with $n \geq 2$. Let $P : S^n \rightarrow \mathbb{R}P^n$ be the standard covering. Define the covering maps $f : \Sigma \rightarrow \mathcal{S} \times N'$ given by $f(x, y) = (x, P(y))$ and $g : N \rightarrow \{0\} \times N'$ given by $g(z) = (0, z)$. The immersion $f : \Sigma \rightarrow M$ is not an embedding. The facts $\pi_1(\Sigma) = \{0\}$ and $\pi_1(M) = \pi_1(\mathbb{R}P^n) = \mathbb{Z}_2$ imply that f_*^1 is not surjective (see Remark 1.2-(iv)).

Let S be an embedded submanifold of a Riemannian manifold M . Let \mathcal{N}_S be the normal bundle of S . We recall that an open subset $W \subset M$ is an ϵ -tubular neighborhood of S if $W = \exp^\perp(\widetilde{W})$, where $\widetilde{W} = \{(x, v) \in \mathcal{N}_S \mid |v| < \epsilon\}$ and the restriction $\exp^\perp|_{\widetilde{W}}$ is a diffeomorphism. Similarly we could define a *closed* ϵ -tubular neighborhood. More generally we could define:

Definition 1.1. Let V be a subset of M that contains S . We say that V is a *tubular neighborhood* of S if $V = \exp^\perp(\widetilde{V})$, where $\exp^\perp|_{\widetilde{V}}$ is a diffeomorphism and \widetilde{V} is a (possibly with boundary) submanifold of \mathcal{N}_S with maximal dimension and with the following property: if $(p, v) \in \widetilde{V}$ then $(p, tv) \in \widetilde{V}$ for all $t \in [0, 1]$.

Comparing with Theorem 1.1, the next result consider the assumption that M is simply-connected instead the assumption that Σ has finite fundamental group. By Remark 1.1, the simply-connectedness of M implies that N is simply-connected and g is an embedding.

Theorem 1.2. *Let $f : \Sigma^{m-1} \rightarrow M^m$ be a totally geodesic immersion of a closed manifold Σ in a complete simply-connected Riemannian manifold M and $g : N^n \rightarrow M^m$ a complete isometric embedding without focal points. Then N is compact and the following conclusions hold:*

- (1) *if $m - n = 1$ then Σ is diffeomorphic to N , and f is an embedding and a homotopy equivalence;*
- (2) *if $m - n \geq 2$ then Σ covers the unit normal bundle \mathcal{N}_g^1 of g and $f(\Sigma) \cap g(N) = \emptyset$; if $m - n = 2$ then $\pi_1(\Sigma)$ is cyclic, and $f_*^i : \pi_i(\Sigma) \rightarrow \pi_i(M)$ is a monomorphism for $i = 2$ and an isomorphism for $i \geq 3$;*
- (3) *if $m - n \geq 3$ then the following statements hold:*
 - (a) *f is an $(m - n - 1)$ -connected embedding;*
 - (b) *Σ is simply-connected and diffeomorphic to \mathcal{N}_g^1 ;*
 - (c) *$M - f(\Sigma) = A \cup B$ (disjoint union) where the closure \bar{A} is a compact tubular neighborhood of $g(N)$ with smooth boundary $f(\Sigma)$, and \bar{B} is an unbounded smooth manifold with boundary $f(\Sigma)$.*

Remark 1.3. Theorem 1.2 shows that if $m - n \neq 2$ then f is an embedding and Σ is simply-connected. However, in the case $m - n = 2$ both conditions may fail. In fact, if we consider in Example 1.3 the special case that $\Sigma = SU_2 \cong S^3$ and $f : \Sigma \rightarrow \Sigma' = T_{\frac{\pi}{2}}S^2$ is the universal covering, we see that the map $f : \Sigma \rightarrow M = TS^2$ is not an embedding. Still in Example 1.3, the case that $\Sigma = \Sigma' = T_{\frac{\pi}{2}}S^2$ we see that Σ is not simply-connected, showing that the simply-connectedness of f may not occur in codimension two.

For the next result we need the following definition.

Definition 1.2. Let S and X be contained in a Riemannian manifold M , where S is an embedded submanifold. We say that X is a normal graph over S if there

exists a homeomorphism $h : S \rightarrow X$ such that for any point $x \in S$ there exists a unique normal geodesic which starts at x orthogonally to S and ends at $h(x)$.

Comparing with Theorem 1.2, the next result replaces the assumption that Σ is compact by the condition that Σ is properly embedded in M with $g(N) \cap \Sigma = \emptyset$. No additional codimension conditions are needed.

Theorem 1.3. *Let Σ be a properly embedded totally geodesic hypersurface in a complete simply-connected manifold M . Let $g : N \rightarrow M$ be a complete isometric embedding without focal points with $g(N) \cap \Sigma = \emptyset$. Then we have:*

- (1) *for any $\epsilon > 0$, the hypersurface Σ is a normal graph over an open subset of the boundary of the closed ϵ -tubular neighborhood of $g(N)$;*
- (2) *$M - \Sigma = A \cup B$ (disjoint union), where the closure \bar{A} is a (possibly unbounded) tubular neighborhood of $g(N)$ with smooth boundary Σ and \bar{B} is an unbounded smooth manifold with boundary Σ ;*
- (3) *for each point $x \in \bar{B}$, the unique normal geodesic which starts orthogonally at $g(N)$ and ends at x intersects Σ transversely at a unique point.*

Theorem B shows that the hypothesis that $\exp^\perp : \mathcal{N}_g \rightarrow M$ is a covering map is weaker than the assumption that M and N are complete with g free of focal points. The next result presents a situation where this two assumptions are equivalent.

Theorem 1.4. *Let $f : \Sigma^{m-1} \rightarrow M^m$ be a totally geodesic immersion of a closed manifold Σ in a Riemannian manifold M . Let $g : N^n \rightarrow M^m$ be an immersion. Assume that one of the following conditions holds:*

- (i) *$\exp^\perp : \mathcal{N}_g \rightarrow M$ is a diffeomorphism;*
- (ii) *$\exp^\perp : \mathcal{N}_g \rightarrow M$ is a covering map and Σ has finite fundamental group.*

Then M is complete and N is compact, hence Theorem 1.1 applies if Item (ii) is satisfied. If Item (i) holds we have that:

- (1) *if $m - n = 1$ then Σ covers N , the map $f_*^i : \pi_i(\Sigma) \rightarrow \pi_i(M)$ is a monomorphism for $i = 1$ and an isomorphism for $i \geq 2$;*
- (2) *if $m - n \geq 2$ then Σ covers the unit normal bundle of g , $f(\Sigma) \cap g(N) = \emptyset$ and it holds that:*
 - (a) *if f is an embedding then f is $(m - n - 1)$ -connected and $M - f(\Sigma) = A \cup B$ (disjoint union) where the closure \bar{A} is a compact tubular neighborhood of $g(N)$ with boundary $f(\Sigma)$, and \bar{B} is an unbounded manifold with boundary $f(\Sigma)$;*
 - (b) *if $m - n = 2$ then $f_*^i : \pi_i(\Sigma) \rightarrow \pi_i(M)$ is a monomorphism for $i = 2$ and an isomorphism for $i \geq 3$;*
- (3) *if $m - n \geq 3$ then $f_*^i : \pi_i(\Sigma) \rightarrow \pi_i(M)$ is a monomorphism for $i = 1$, an isomorphism for $2 \leq i \leq m - n - 2$ and an epimorphism for $i = m - n - 1$.*

The following example illustrates Theorem 1.4-(1).

Example 1.6. We consider the complete flat Moebius strip

$$M = ([-1, 1] \times \mathbb{R}) / \sim, \text{ where } (-1, t) \sim (1, -t) \text{ for all } t \in \mathbb{R}.$$

Denote by $\bar{\alpha} \in M$ the class of α . Take

$$\Sigma = \left\{ \overline{(x, t)} \in M \mid x \in [-1, 1], t = 1 \text{ or } t = -1 \right\}$$

and $N = \left\{ \overline{(x, 0)} \in M \mid x \in [-1, 1] \right\}$ and let $f : \Sigma \rightarrow M$ and $g : N \rightarrow M$ be the inclusion maps. It is easy to see that f is totally geodesic and that $\exp^\perp : \mathcal{N}_g \rightarrow M$ is a diffeomorphism.

One of the conclusions of Theorem 1.1 is that M and N have finite fundamental groups. Thus one could ask if this conclusion should also be obtained from Condition (i) of Theorem 1.4. The next example shows that this is not the case.

Example 1.7. Let N' be a closed Riemannian manifold with infinite fundamental group and M the Riemannian product $M = N' \times V$, where $V = (\mathbb{R}^k, ds^2)$, where ds^2 is the metric introduced in Example 1.4. Let $\mathcal{S} = S^{k-1} \subset V$ be the totally geodesic sphere. Let $g : N = N' \times \{0\} \rightarrow M$ and $f : \Sigma = N' \times \mathcal{S} \rightarrow M$ be the inclusion maps. We see that f is totally geodesic and g is free of focal points, hence Theorem 1.4 applies. It holds that M and N have infinite fundamental groups.

Theorem 1.4 suggests that Theorems 1.1, 1.2 and 1.3 could be rewritten in technical more general versions (see Theorems 2.1, 3.2 and 3.3, where we also generalize the totally geodesic condition).

The rest of this paper is organized as follows. In section 2 we prove Theorem 1.3 and in section 3 we prove Theorems 1.1, 1.2 and 1.4. In section 4 we present proofs for facts present in some examples in the introduction.

Remark 1.4. Since this paper uses Proposition 4.1 in [MZ], the first author would like to use this occasion to inform that it came to his knowledge that Theorem A in [MZ] is implied by a more general result present in the doctor thesis of Florêncio F. Guimarães.

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2. Proof of Theorem 1.3

Theorem 1.3 follows from Theorem B and the next technical general theorem (compare with Theorem 1.2 in [MM]):

Theorem 2.1. *Let Σ be a properly embedded hypersurface in a Riemannian manifold M . Let $g : N \rightarrow M$ be an embedding such that $\exp^\perp : \mathcal{N}_g \rightarrow M$ is a diffeomorphism. Assume that $g(N) \cap \Sigma = \emptyset$ and that the normal geodesics tangent to Σ do not intersect $g(N)$ orthogonally. Then we have:*

- (1) *given $\epsilon > 0$, the hypersurface Σ is a normal graph over an open subset of the boundary of the closed ϵ -tubular neighborhood of $g(N)$;*
- (2) *$M - \Sigma = A \cup B$ (disjoint union), where \bar{A} is a (possibly unbounded) tubular neighborhood of $g(N)$ with smooth boundary Σ and \bar{B} is an unbounded smooth manifold with boundary Σ ;*
- (3) *for each point $x \in \bar{B}$ the unique normal geodesic $\gamma_x : [0, +\infty) \rightarrow M$ which starts orthogonally at $g(N)$ and passes through x intersects Σ transversely at a unique point.*

It is a little surprising that the hypotheses of completeness of M and N are not needed in Theorem 2.1. What compensates this weakness is the fact that for each point x in $M - g(N)$ there exist suitable neighborhoods W of x with the property that $\gamma_y([0, +\infty)) \subset W$, for all $y \in W$. Thus we can use local arguments to explore the fact that Σ is properly embedded.

The two simple examples bellow illustrate this theorem.

Example 2.1. Consider the surface $\Sigma \subset \mathbb{R}^3$ of equation $z = \frac{1}{x^2+y^2}$. Notice that Σ is properly embedded in $M = \{(x, y, z) \in \mathbb{R}^3 \mid z > 0\}$. Let $g : N = \{(0, 0, z) \in \mathbb{R}^3 \mid z > 0\} \rightarrow M$ be the inclusion map. Notice that $\exp^\perp : \mathcal{N}_g \rightarrow M$ is a diffeomorphism and the normal geodesics tangent to Σ do not intersect N orthogonally. Thus Theorem 2.1 applies although M and N are not complete.

Example 2.2. Consider the surface $\Sigma \subset \mathbb{R}^3$ of equation $z = \frac{1}{1-x^2-y^2}$, with $x^2 + y^2 < 1$. The surface Σ is properly embedded in $M = \mathbb{R}^3$. Let N be the xy -plane and $g : N \rightarrow M$ the inclusion map. Theorem 2.1 applies. Notice that

$$B = \left\{ (x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 < 1 \text{ and } z > \frac{1}{1-x^2-y^2} \right\},$$

and that the complement $A = \mathbb{R}^3 - \bar{B}$ is a tubular neighborhood of N in the sense of Definition 1.1.

Before proving Theorem 2.1 we would like to present some notations and a very simple result that will be used here and in other places in this paper. Let $g : N^n \rightarrow M^m$ be an immersion such that $\exp^\perp : \mathcal{N}_g \rightarrow M$ is a diffeomorphism. Then g is an embedding and the natural projection $\rho : M \rightarrow g(N)$ given by $\rho(\exp^\perp(q, v)) = g(q)$ is a fiber bundle. From the fact that \exp^\perp is a diffeomorphism it follows easily that ρ is a homotopy equivalence between M and $g(N)$. Furthermore, for each point $x \in M - g(N)$ there exists a unique normal geodesic $\gamma_x : [0, +\infty) \rightarrow M$ containing x which intersects $g(N)$ orthogonally at $t = 0$ satisfying $\gamma_x(d_x) = x$ for some $d_x > 0$. We should notice that we don't know here if d_x is the distance $d(x, g(N))$ (remember that we are not assuming that M and N are complete). Fix $\epsilon > 0$ and let S_ϵ be the boundary of the ϵ -tubular neighborhood of $g(N)$. Let $j : S_\epsilon \rightarrow M$ be the inclusion map.

Lemma 2.1. Let $g : N^n \rightarrow M^m$ be an embedding such that $\exp^\perp : \mathcal{N}_g \rightarrow M$ is a diffeomorphism. Assume notations above. Let $f : \Sigma^{m-1} \rightarrow M^m$ be an immersion such that the normal geodesics tangent to $f(\Sigma)$ do not intersect $g(N)$ orthogonally. Then we have:

- (a) The map $\xi : M \rightarrow \mathbb{R}$ given by $\xi(x) = d_x$ is continuous on M and smooth on $M - g(N)$;
- (b) If $f(\Sigma) \cap g(N) = \emptyset$ then the map $F : \Sigma \rightarrow S_\epsilon$ given by $F(p) = \gamma_{f(p)}(\epsilon)$ is a local diffeomorphism and f is homotopic to $(j \circ F)$;
- (c) If $m - n = 1$ then the map $G : \Sigma \rightarrow g(N)$ given by $G = \rho \circ f$ is a local diffeomorphism.

Proof. We first prove Item (a). For $x \in M$, we write $(p, v) = (\exp^\perp)^{-1}(x)$ and define $P : x \mapsto v$. Since $d_x = |P(x)|$, we conclude that the map ξ is continuous on M and smooth on $M - g(N)$.

To prove Item (b) define the vector field X on $M - g(N)$ given by $X(x) = \gamma'_x(d_x)$. Fix a small open subset $U \subset \Sigma$ such that $f|_U : U \rightarrow M$ is an embedding. The orbits of X are orthogonal to S_ϵ and, by hypothesis, transversal to $f(U)$. Thus, reducing U if necessary, it is not difficult to see that the flow of X gives a standard diffeomorphism $\varphi : f(U) \rightarrow V$ where V is a small open subset of S_ϵ . Note that $F|_U = \varphi \circ f|_U$, hence $F|_U : U \rightarrow V$ is a diffeomorphism, hence F is a local

diffeomorphism. The map $H : [0, 1] \times \Sigma \rightarrow M$ given by

$$H(t, x) = \gamma_{f(x)}((1-t)\epsilon + td_{f(x)})$$

provides a homotopy between f and $j \circ F$.

To prove Item (c), notice that $\rho : M \rightarrow g(N)$ given by $\rho(\exp^\perp(q, v)) = g(q)$ is a fiber bundle over $g(N)$ such that, given $z \in M$, the fiber containing z is the image of a geodesic $\beta_z : \mathbb{R} \rightarrow M$ which intersects $g(N)$ orthogonally and satisfies $\beta_z(0) = z$. If further there exists $x \in \Sigma$ such that $z = f(x)$, we know by hypothesis that $\beta'_z(0) \notin df_x(T_x \Sigma)$, hence f is transversal to β_z . Thus the map $G : \Sigma \rightarrow g(N)$ given by $G(x) = \rho \circ f(x)$ is a submersion, hence a local diffeomorphism. \square

Proof of Theorem 2.1. Since $\exp^\perp : \mathcal{N}_g \rightarrow M$ is a diffeomorphism and $\Sigma \cap g(N) = \emptyset$, Lemma 2.1 implies that the map $F : \Sigma \rightarrow S_\epsilon$ given by $F(p) = \gamma_p(\epsilon)$ is a local diffeomorphism onto its open image $F(\Sigma) \subset S_\epsilon$. To prove that F is a diffeomorphism it is sufficient to show that F is injective. In order to show this fact we define the set

$$\mathcal{C} = \{p \in \Sigma \mid \text{the cardinality } \#(\gamma_p([0, d_p]) \cap \Sigma) = 1\}.$$

We just need to prove that $\mathcal{C} = \Sigma$.

Claim 2.1. $\mathcal{C} \neq \emptyset$.

In fact we take $p \in \Sigma$. Since Σ is properly embedded it follows that $\gamma_p([0, d_p]) \cap \Sigma$ is a compact set. By using the facts that γ_p intersects Σ transversely and that Σ is properly embedded, we obtain that $\gamma_p([0, d_p]) \cap \Sigma$ is a discrete set, and thus it is finite. Thus we write

$$\gamma_p([0, d_p]) \cap \Sigma = \{\gamma_p(t_1), \dots, \gamma_p(t_k)\}$$

with $t_1 < \dots < t_k = d_p$. Notice that $t_1 > 0$ since $\Sigma \cap g(N) = \emptyset$. Set $p_1 = \gamma_p(t_1)$. Since $\gamma_{p_1}|_{[0, d_{p_1}]} = \gamma_p|_{[0, t_1]}$, we have that $\#(\Sigma \cap \gamma_{p_1}([0, d_{p_1}])) = 1$, hence $p_1 \in \mathcal{C}$, which concludes the proof of this Claim.

Claim 2.2. $\Sigma - \mathcal{C}$ is open as a subset of Σ .

To prove this take $x_1 \in \Sigma - \mathcal{C}$. So there exists $x_2 \in \Sigma$ with $x_2 \neq x_1$ and $x_2 = \gamma_{x_1}(t)$ for some $0 < t < d_{x_1}$. In particular $F(x_1) = F(x_2) = \gamma_{x_1}(\epsilon)$. Since F is a local diffeomorphism, there exist disjoint neighborhoods of x_1 and x_2 in Σ mapped by F onto the same neighborhood of $\gamma_{x_1}(\epsilon)$ in S_ϵ . Thus we conclude that $\Sigma - \mathcal{C}$ is open in Σ .

Claim 2.3. $\Sigma - \mathcal{C}$ is closed as a subset of Σ .

In fact take a sequence $x_k \rightarrow x \in \Sigma$ with $x_k \in \Sigma - \mathcal{C}$. By Lemma 2.1 we have that $d_{x_k} \rightarrow d_x$. Since $\gamma_y(0) = \rho(y)$, where $\rho : M \rightarrow g(N)$ given by $\rho(\exp^\perp(q, v)) = g(q)$ is the natural projection, we have by continuity of ρ that $\gamma_{x_k}(0) \rightarrow \gamma_x(0)$. Since Σ is properly embedded there exists an open neighborhood U of x in M such that the intersection $\Sigma \cap U$ is a topological disk and the restriction $F|_{U \cap \Sigma}$ is a diffeomorphism onto its open image. By passing to a subsequence we may assume that $\gamma'_{x_k}(0) \rightarrow v$. Since $x_k = \gamma_{x_k}(d_{x_k}) = \exp^\perp(\gamma_{x_k}(0), d_{x_k} \gamma'_{x_k}(0))$, by taking limits we obtain that $x = \exp^\perp(\gamma_x(0), d_x v)$. By the other hand we have that $x = \gamma_x(d_x) = \exp^\perp(\gamma_x(0), d_x \gamma'_x(0))$, thus the injectivity of \exp^\perp implies that $v = \gamma'_x(0)$. Since $\gamma_{x_k}(0) \rightarrow \gamma_x(0)$ and $\gamma'_{x_k}(0) \rightarrow \gamma'_x(0)$ we obtain that $\gamma_{x_k} \rightarrow \gamma_x$ uniformly on compact sets. Since $x_k \notin \mathcal{C}$ there exists a point $y_k \neq x_k$ with $y_k \in \Sigma$

and $y_k = \gamma_{x_k}(t_k)$ with $0 < t_k < d_{x_k}$. The sequence (t_k) is bounded since (d_{x_k}) converges. Again by passing to a subsequence we can suppose that t_k converges to some $t_0 \in [0, d_x]$, hence $y_k = \gamma_{x_k}(t_k) \rightarrow x_0 = \gamma_x(t_0)$, which belongs to Σ because Σ is properly embedded. For large k the point $x_k \in U$. Since $F|_{U \cap \Sigma}$ is injective and $F(x_k) = F(y_k)$ we have that $y_k \notin U$ for large k , hence $x_0 = \gamma_x(t_0) \neq x$. Thus $t_0 < d_x$, hence $x \in \Sigma - \mathcal{C}$. This concludes the proof of Claim 2.3.

By the connectedness of Σ we conclude that $\mathcal{C} = \Sigma$ which proves the following

Claim 2.4. $F : \Sigma \rightarrow F(\Sigma)$ is a diffeomorphism.

From Claim 2.4 we have that Σ is a normal graph over the open subset $F(\Sigma) \subset S_\epsilon$ (see Definition 1.2). This proves Item (1) in Theorem 2.1.

Now we will prove that Σ is the boundary of a tubular neighborhood of $g(N)$. Define the set

$$A = \{x \in M \mid \gamma_x([0, d_x]) \cap \Sigma = \emptyset\}.$$

Claim 2.5. A is an open subset of M .

In fact it suffices to prove that $M - A$ is closed. Consider a sequence $x_k \in M - A$ such that $x_k \rightarrow x$. Thus for each k there exists $y_k \in \gamma_{x_k}([0, d_{x_k}]) \cap \Sigma$ and we write $y_k = \gamma_{x_k}(t_k)$, with $0 < t_k \leq d_{x_k}$. As in the proof of Claim 2.3, we obtain by passing to a subsequence that $t_k \rightarrow t_0 \in [0, d_x]$, hence $y_k = \gamma_{x_k}(t_k) \rightarrow y = \gamma_x(t_0)$. Since Σ is properly embedded we have that $y \in \Sigma$, hence $y \in \gamma_x([0, d_x]) \cap \Sigma$. Thus we conclude that $x \in M - A$, hence $M - A$ is closed.

Claim 2.6. $\bar{A} - A = \Sigma$.

In fact, since $\mathcal{C} = \Sigma$, given any $p \in \Sigma$ we have that $\gamma_p([0, d_p]) - \{p\} \subset A$. Thus $\Sigma \subset \bar{A}$. Clearly $\Sigma \cap A = \emptyset$ hence $\Sigma \subset \bar{A} - A$. Now take $p \in \bar{A} - A$ and assume by contradiction that $p \notin \Sigma$. Since $p \notin A$ we have $\gamma_p([0, d_p]) \cap \Sigma \neq \emptyset$. Since the map F is injective we obtain that $\#(\gamma_p([0, d_p]) \cap \Sigma) \leq 1$, hence $\gamma_p([0, d_p])$ intersects Σ transversely at a unique point $q \in \Sigma$. Thus it holds that $\gamma_q([0, +\infty)) = \gamma_p([0, +\infty))$ and $d_q < d_p$. Let U be a small neighborhood of q in Σ . For a small $0 < \delta < d_p - d_q$ consider the set

$$W = \{\gamma_x(t) \mid x \in U \text{ and } d_p - \delta < t < d_p + \delta\}.$$

Notice that $p = \gamma_q(d_p)$, hence $p \in W$. Since \exp^\perp is a diffeomorphism the set W is an open neighborhood of p . By taking U sufficiently small and using that $d_q < d_p - \delta$, we obtain by continuity that $d_x < d_p - \delta$ for all $x \in U$. Now take $y \in W$. Then there exists $x \in U$ and $d_p - \delta < t < d_p + \delta$ such that $y = \gamma_x(t) = \gamma_x(d_y)$. Since $d_x < d_p - \delta < t = d_y$ we have that $x \in \gamma_y([0, d_y]) \cap \Sigma$, hence $y \notin A$. Thus we have that $W \subset M - A$ which contradicts the fact that $p \in \bar{A}$. This concludes the proof of Claim 2.6.

Let us prove that \bar{A} is a manifold with smooth boundary Σ . Take a point $p \in \Sigma$. For a small neighborhood V of p in Σ and small $\epsilon > 0$, consider the set

$$\Omega = \{\gamma_x(t) \mid x \in V; d_x - \epsilon < t \leq d_x\}.$$

By Lemma 2.1 we have that d_x depends smoothly on x on $M - g(N)$. Since \exp^\perp is a diffeomorphism we conclude that Ω is a parameterized neighborhood of p in \bar{A} . Furthermore we have

$$\Omega = \{\gamma_x(t) \mid x \in V; d_x - \epsilon < t < d_x\} \cup \{\gamma_x(t) \mid x \in V; t = d_x\} = (A \cap \Omega) \cup (\Sigma \cap \Omega).$$

Thus \bar{A} is a smooth manifold with boundary Σ . We have proved that $\gamma_x([0, d_x]) \subset A$ for all $x \in A$ and $\gamma_x([0, d_x]) \subset \bar{A}$ for all $x \in \bar{A}$. Thus we conclude that A and \bar{A} are tubular neighborhoods of $g(N)$.

To see that Σ disconnects M consider $B = M - \bar{A}$. Thus we have $M = \bar{A} \cup B = A \cup \Sigma \cup B$, where the unions are disjoint. In particular we have $M - \Sigma = A \cup B$. Fix $p \in \bar{B}$. Since $p \notin A$ we conclude by the proof of Claim 2.6 that γ_p intersects Σ transversely at a unique point. Thus we have that

$$\bar{B} = \{p \in M \mid \#(\gamma_p([0, d_p]) \cap \Sigma) = 1\} = \{\gamma_x(t) \mid x \in \Sigma \text{ and } t \geq d_x\},$$

which proves that B is a connected non-compact manifold with boundary Σ . Items (2) and (3) in Theorem 2.1 are proved. \square

3. Proof of Theorems 1.1, 1.2 and 1.4

To prove Theorems 1.1, 1.2 and 1.4 we first need to show the following simple lemmas.

Lemma 3.1. Let $f : \Sigma \rightarrow M$ be a totally geodesic immersion of a closed manifold Σ in a Riemannian manifold M and $g : N \rightarrow M$ an immersion. Assume that either

- (a) $\exp^\perp : \mathcal{N}_g \rightarrow M$ is a diffeomorphism; or
- (b) $\exp^\perp : \mathcal{N}_g \rightarrow M$ is a covering map and Σ has finite fundamental group.

Then the normal geodesics of M tangent to $f(\Sigma)$ do not intersect $g(N)$ orthogonally.

Proof. First, we will assume that (a) is satisfied. Assume by contradiction that there exists a normal geodesic $\gamma : [0, +\infty) \rightarrow M$ that starts orthogonally to $g(N)$ and is tangent to $f(\Sigma)$. Since f is totally geodesic we conclude that $\gamma([0, +\infty)) \subset f(\Sigma)$, hence γ is bounded, since Σ is compact. We write $\gamma(t) = \exp^\perp(q, tv)$, $t \geq 0$, for some $(q, v) \in \mathcal{N}_g$ with $|v| = 1$. Since \exp^\perp is a diffeomorphism we have that γ is unbounded, which gives us a contradiction.

Now we assume (b). Consider on \mathcal{N}_g the metric induced by $\exp^\perp = \exp_g^\perp$ and define $\hat{g} : N \rightarrow \mathcal{N}_g$ given by $\hat{g}(x) = (x, 0)$. Let $\nu : \hat{\Sigma} \rightarrow \Sigma$ be the universal covering of Σ with the induced metric. By the Fundamental Lifting Theorem, f admits a lifting $\hat{f} : \hat{\Sigma} \rightarrow \mathcal{N}_g$ such that the diagram below is commutative.

$$(1) \quad \begin{array}{ccccc} \hat{\Sigma} & \xrightarrow{\hat{f}} & \mathcal{N}_g & & \\ \nu \downarrow & & \exp_g^\perp \downarrow & \nwarrow \hat{g} & \\ \Sigma & \xrightarrow{f} & M & \xleftarrow{g} & N \end{array}$$

Since Σ is compact with finite fundamental group it follows that $\hat{\Sigma}$ is compact. Note that \hat{f} is totally geodesic. Any normal geodesic γ on M starting orthogonally from $g(N)$ is lifted by \exp_g^\perp to a curve $\hat{\gamma} = \hat{\gamma}_{(q,v)} : [0, +\infty) \rightarrow \mathcal{N}_g$ given by $\hat{\gamma}(t) = (q, tv)$, for some $(q, v) \in \mathcal{N}_g$ with $|v| = 1$. Since we are considering on \mathcal{N}_g the metric induced by \exp_g^\perp , we have that the curves $\hat{\gamma}_{(q,v)}$ are the normal geodesics of \mathcal{N}_g that start orthogonally from $\hat{g}(N) = N \times \{0\}$, hence $\exp_g^\perp : \mathcal{N}_g \rightarrow M$ is a diffeomorphism. Thus we may use Item (a) to conclude that the normal geodesics tangent to $\hat{f}(\hat{\Sigma})$ do not intersect $\hat{g}(N)$ orthogonally. Since \exp_g^\perp is a local isometry we conclude that the normal geodesics tangent to $f(\Sigma)$ do not intersect $g(N)$ orthogonally. \square

Lemma 3.2. Let $f : \Sigma \rightarrow M$ and $g : N \rightarrow M$ be immersions in a Riemannian manifold M with dimension $\dim(\Sigma) > \dim(N)$. Assume that the normal geodesics of M tangent to $f(\Sigma)$ do not intersect $g(N)$ orthogonally. Then $f(\Sigma) \cap g(N) = \emptyset$.

Proof. Assume by contradiction that $f(\Sigma) \cap g(N) \neq \emptyset$. Then there exist $p \in \Sigma$ and $q \in N$ with $f(p) = g(q)$. Set

$$V = (dg_q(T_q N))^\perp \cap (df_p(T_p \Sigma)),$$

where df_p denotes the derivative of f at p and $T_p \Sigma$ the tangent space. Then we have:

$$\begin{aligned} \dim(V) &\geq \dim(dg_q(T_q N))^\perp + \dim(df_p(T_p \Sigma)) - \dim(M) \\ &= (\dim(M) - \dim(N)) + \dim(\Sigma) - \dim(M) \geq 1. \end{aligned}$$

Thus we can take $w \in V$ with $|w| = 1$. Consider the geodesic $\gamma : \mathbb{R} \rightarrow M$ satisfying $\gamma(0) = g(q)$ and $\gamma'(0) = w$. Thus γ is a geodesic tangent to $f(\Sigma)$ and orthogonal to $g(N)$, which is a contradiction. \square

From Lemmas 3.1 and 3.2 we obtain the following

Corollary 3.1. Let $f : \Sigma \rightarrow M$ be a totally geodesic immersion of a closed manifold Σ in a Riemannian manifold M and $g : N \rightarrow M$ an immersion with dimension $\dim(\Sigma) > \dim(N)$. Assume that either

- (a) $\exp^\perp : \mathcal{N}_g \rightarrow M$ is a diffeomorphism; or
- (b) $\exp^\perp : \mathcal{N}_g \rightarrow M$ is a covering map and Σ has finite fundamental group.

Then $f(\Sigma) \cap g(N) = \emptyset$.

Remark 3.1. Example 1.1 shows that the conclusion of Corollary 3.1 fails if $\dim(\Sigma) = \dim(N)$.

The next result will be important to prove Theorem 1.1, 1.2 and 1.4.

Theorem 3.1. Let $f : \Sigma^{m-1} \rightarrow M^m$ be an immersion of a closed manifold Σ in a Riemannian manifold M . Let $g : N \rightarrow M$ be an embedding such that $\exp^\perp : \mathcal{N}_g \rightarrow M$ is a diffeomorphism. Assume that the normal geodesics of M tangent to $f(\Sigma)$ do not intersect $g(N)$ orthogonally. Then M is complete, N is compact and the following conditions hold:

- (1) if $m - n = 1$ then Σ covers N , the map $f_*^i : \pi_i(\Sigma) \rightarrow \pi_i(M)$ is a monomorphism for $i = 1$ and an isomorphism for $i \geq 2$;
- (2) if $m - n \geq 2$ then Σ covers the unit normal bundle \mathcal{N}_g^1 , $f(\Sigma) \cap g(N) = \emptyset$ and we have:
 - (a) if f is an embedding then f is $(m - n - 1)$ -connected and $M - f(\Sigma) = A \cup B$ (disjoint union) where the closure \bar{A} is a compact tubular neighborhood of $g(N)$ with boundary $f(\Sigma)$, and \bar{B} is an unbounded manifold with boundary $f(\Sigma)$;
 - (b) if $m - n = 2$ then $f_*^i : \pi_i(\Sigma) \rightarrow \pi_i(M)$ is a monomorphism for $i = 2$ and an isomorphism for $i \geq 3$;
- (3) if $m - n \geq 3$ then $f_*^i : \pi_i(\Sigma) \rightarrow \pi_i(M)$ is a monomorphism for $i = 1$, an isomorphism for $2 \leq i \leq m - n - 2$ and an epimorphism for $i = m - n - 1$.

To prove Theorem 3.1 we will need to use the following result which follows from the proof of Proposition 4.1 in [MZ]:

Proposition 3.1. *Let M be a Riemannian manifold and $g : N \rightarrow M$ an immersion. Assume that for any point $p \in M$ there exists $q \in N$ such that the distance $d(p, g(N)) = d(p, g(q))$. Assume further that \exp^\perp is defined on all \mathcal{N}_g . Then M is complete.*

Corollary 3.2. *Let $g : N \rightarrow M$ be an embedding such that $\exp^\perp : \mathcal{N}_g \rightarrow M$ is a diffeomorphism. If N is compact then M is complete.*

Proof of Theorem 3.1. We first prove that M is complete and N is compact. By Corollary 3.2 it suffices to prove that N is compact. We first consider the case $m - n = 1$. By Lemma 2.1-(c) the map $G : \Sigma \rightarrow g(N)$, given by $G = \rho \circ f$ where $\rho : M \rightarrow g(N)$ is the natural projection, is a local diffeomorphism. By compactness of Σ and connectedness of $g(N)$ we have that $G(\Sigma) = g(N)$, hence N is compact since g is an embedding. Now we consider the case that $m - n \geq 2$. By Lemma 3.2 we obtain that $f(\Sigma) \cap g(N) = \emptyset$. Thus by Lemma 2.1-(b) the map $F : \Sigma \rightarrow S_\epsilon$ given by $F(p) = \gamma_{f(p)}(\epsilon)$ is a local diffeomorphism. From the connectedness of S_ϵ and the compactness of Σ we conclude that $F(\Sigma) = S_\epsilon$. This implies that N is compact since $S_\epsilon = F(\Sigma)$ is a compact bundle over N .

To prove Item (1) we assume that $m - n = 1$. We obtain that $G = \rho \circ f : \Sigma \rightarrow g(N)$ is a covering map from the following facts: the map G is a local diffeomorphism; the manifold Σ is compact; N is connected. Since g is an embedding we conclude that Σ covers N . We know that ρ is a homotopy equivalence between M and $g(N)$, hence $\rho_*^i : \pi_i(M) \rightarrow \pi_i(g(N))$ is an isomorphism for all i . The fact that G is a covering map implies that $G_*^i : \pi_i(\Sigma) \rightarrow \pi_i(g(N))$ is a monomorphism for $i = 1$ and an isomorphism for $i \geq 2$. From the equality $G_*^i = \rho_*^i \circ f_*^i$, we conclude that $f_*^i : \pi_i(\Sigma) \rightarrow \pi_i(M)$ is a monomorphism for $i = 1$ and an isomorphism for $i \geq 2$. We concluded the proof of Item (1).

From now on we assume that $m - n \geq 2$. We already saw that $F : \Sigma \rightarrow S_\epsilon$ is a local diffeomorphism onto S_ϵ , which is diffeomorphic to \mathcal{N}_g^1 . From the compactness of Σ and the connectedness of S_ϵ it follows that F is a covering map. We conclude that Σ covers \mathcal{N}_g^1 . From Lemma 3.2 we have that $f(\Sigma) \cap g(N) = \emptyset$. In order to prove Item (2)-(a) we assume that f is an embedding. By Claim 2.4 in the proof of Theorem 2.1 we have that F is a diffeomorphism onto its image $F(\Sigma) = S_\epsilon$. Since \exp^\perp is a diffeomorphism we have that the inclusion $j : S_\epsilon \rightarrow M$ is $(m - n - 1)$ -connected. Thus by the fact that $F : \Sigma \rightarrow S_\epsilon$ is a diffeomorphism we conclude that $(j \circ F)$ is $(m - n - 1)$ -connected. By Lemma 2.1-(b), the map $j \circ F : \Sigma \rightarrow M$ is homotopic to $f : \Sigma \rightarrow M$, hence f is $(m - n - 1)$ -connected. Again from Theorem 2.1 we obtain that $M - f(\Sigma) = A \cup B$ (disjoint union) where the closure \bar{A} is a tubular neighborhood of $g(N)$ with boundary $f(\Sigma)$, and \bar{B} is an unbounded manifold with boundary $f(\Sigma)$. Since $f(\Sigma)$ and $g(N)$ are compact we conclude that \bar{A} is compact. Item (2)-(a) is proved.

Let $\iota : g(N) \rightarrow M$ be the inclusion map. We know that $\iota \circ \rho$ is homotopic to the identity map on M . Thus for all i we have that $f_*^i = (\iota \circ \rho \circ f)_*^i = (\iota \circ \rho|_{S_\epsilon} \circ F)_*^i$, hence

$$(2) \quad (\rho|_{S_\epsilon} \circ F)_*^i = (\iota_*^i)^{-1} \circ f_*^i.$$

In order to prove Item (2)-(b) we assume that $m - n = 2$. For $i \geq 2$ we consider the circle bundle $S^1 \rightarrow S_\epsilon \rightarrow g(N)$ to obtain the exact sequence

$$0 = \pi_i(S^1) \longrightarrow \pi_i(S_\epsilon) \xrightarrow{(\rho|_{S_\epsilon})_*^i} \pi_i(g(N)) \longrightarrow \pi_{i-1}(S^1)$$

This implies that $(\rho|_{S_\epsilon})_*^i$ is a monomorphism for $i = 2$ and an isomorphism for $i \geq 3$. Since $F : \Sigma \rightarrow S_\epsilon$ is a covering map, we have that $F_*^i : \pi_i(\Sigma) \rightarrow \pi_i(S_\epsilon)$ is an isomorphism for $i \geq 2$. Thus we have that $(\rho|_{S_\epsilon} \circ F)_*^i : \pi_i(\Sigma) \rightarrow \pi_i(g(N))$ is a monomorphism for $i = 2$ and an isomorphism for $i \geq 3$. By (2) we conclude that f_*^i is a monomorphism for $i = 2$ and an isomorphism for $i \geq 3$. Item (2)-(b) is proved.

Now we assume that $m - n \geq 3$ in order to prove Item (3). By Lemma 2.1-(b) we have that $j \circ F : \Sigma \rightarrow M$ is homotopic to $f : \Sigma \rightarrow M$. Since $F : \Sigma \rightarrow S_\epsilon$ is a covering map and $j : S_\epsilon \rightarrow M$ is $(m - n - 1)$ -connected we obtain from the equality $f_*^i = j_*^i \circ F_*^i$ that f_*^i is a monomorphism for $i = 1$, an isomorphism for $2 \leq i \leq m - n - 2$ and an epimorphism for $i = m - n - 1$. Item (3) is proved. The proof of Theorem 3.1 is complete. \square

Theorem 1.2 follows from the next result together with Remark 1.1 and Lemma 3.1.

Theorem 3.2. *Let $f : \Sigma^{m-1} \rightarrow M^m$ be an immersion of a closed manifold Σ in a complete simply-connected Riemannian manifold M and $g : N^n \rightarrow M^m$ a complete isometric embedding without focal points. Assume that the normal geodesics of M tangent to $f(\Sigma)$ do not intersect $g(N)$ orthogonally. Then N is compact and the following conclusions hold:*

- (1) *if $m - n = 1$ then Σ is diffeomorphic to N , f is an embedding and a homotopy equivalence;*
- (2) *if $m - n \geq 2$ then $f(\Sigma) \cap g(N) = \emptyset$; if $m - n = 2$ then $\pi_1(\Sigma)$ is cyclic, and $f_*^i : \pi_i(\Sigma) \rightarrow \pi_i(M)$ is a monomorphism for $i = 2$ and an isomorphism for $i \geq 3$;*
- (3) *if $m - n \geq 3$ then the following statements hold:*
 - (a) *f is an $(m - n - 1)$ -connected embedding;*
 - (b) *Σ is simply-connected and diffeomorphic to the unit normal bundle \mathcal{N}_g^1 of g ;*
 - (c) *$M - f(\Sigma) = A \cup B$ (disjoint union) where the closure \bar{A} is a compact tubular neighborhood of $g(N)$ with smooth boundary $f(\Sigma)$, and \bar{B} is an unbounded smooth manifold with boundary $f(\Sigma)$.*

Proof. From Remark 1.1 we have that $\exp^\perp : \mathcal{N}_g \rightarrow M$ is a diffeomorphism, hence N is simply-connected and the hypotheses of Theorem 3.1 hold. This implies that N is compact.

We first assume that $m - n = 1$. From the proof of Item (1) of Theorem 3.1, the map $G = \rho \circ f : \Sigma \rightarrow g(N)$ is a covering map. Since $g(N)$ is simply-connected we conclude that G is a diffeomorphism, hence Σ and N are diffeomorphic and f is an embedding. In particular Σ is simply-connected. From Item (1) of Theorem 3.1 we have that the map $f_*^i : \pi_i(\Sigma) \rightarrow \pi_i(M)$ is an isomorphism for $i \geq 2$. Since $\pi_1(\Sigma) = \pi_1(M) = 0$ we conclude from the Whitehead's Theorem that f is a homotopy equivalence between Σ and M . Item (1) is proved.

From Theorem 3.1, to prove Item (2) we just need to prove that $\pi_1(\Sigma)$ is cyclic under the condition $m - n = 2$. We consider the fibre bundle $S^1 \rightarrow S_\epsilon \rightarrow g(N)$ with the corresponding exact sequence

$$\mathbb{Z} = \pi_1(S^1) \rightarrow \pi_1(S_\epsilon) \rightarrow \pi_1(g(N)) = 0$$

hence $\pi_1(S_\epsilon)$ is cyclic. Since $F : \Sigma \rightarrow S_\epsilon$ is a covering map we have that $F_*^1 : \pi_1(\Sigma) \rightarrow \pi_1(S_\epsilon)$ is injective, hence $\pi_1(\Sigma)$ is cyclic. Item (2) is proved.

Now we assume that $m - n \geq 3$. The map $F : \Sigma \rightarrow S_\epsilon$ given by $F(x) = \gamma_{f(x)}(\epsilon)$ is a covering map. From the fiber bundle $S^k \rightarrow S_\epsilon \rightarrow g(N)$ with $k \geq 2$ and from the fact that N is simply-connected we conclude that S_ϵ is simply-connected, hence F is a diffeomorphism, which implies that Σ is simply-connected, f is an embedding and Σ is diffeomorphic to \mathcal{N}_g^1 . By Item (2)-(a) of Theorem 3.1 we conclude that f is $(m - n - 1)$ -connected and $M - f(\Sigma) = A \cup B$ (disjoint union) where the closure \bar{A} is a compact tubular neighborhood of $g(N)$ with boundary $f(\Sigma)$, and \bar{B} is an unbounded manifold with boundary $f(\Sigma)$, which concludes the proof of Item (3). Theorem 3.2 is proved. \square

To prove the next theorem we will need the following topological lemma. Since we didn't find it in the literature, we will present its simple proof for the sake of completeness.

Lemma 3.3. Let M be a connected metric space, and A, B subsets of M with closures and boundaries connected. Assume that $\bar{A} \cap B \neq \emptyset$, $A \cap \bar{B} \neq \emptyset$, $\partial A \cap \partial B = \emptyset$, $\bar{A} \not\subset B$ and $\bar{B} \not\subset A$. Then $M = \text{int}(A) \cup \text{int}(B)$, where $\text{int}(A)$ denotes the interior of A .

Proof. By hypothesis we have that $\bar{A} \cap B \neq \emptyset$ and $\bar{A} \cap (M - B) \neq \emptyset$. By connectedness of \bar{A} we conclude that $\bar{A} \cap \partial B \neq \emptyset$. Since $\partial A \cap \partial B = \emptyset$ we obtain that

$$\text{int}(A) \cap \partial B = \bar{A} \cap \partial B \neq \emptyset.$$

Thus the set $\text{int}(A) \cap \partial B$ is an open and closed nonempty subset of the connected set ∂B , hence we have that $\text{int}(A) \cap \partial B = \partial B$. Thus we obtain that $\partial B \subset \text{int}(A)$. Similarly we can prove that $\partial A \subset \text{int}(B)$. We conclude that

$$\bar{A} \cup \bar{B} = (\text{int}(A) \cup \partial A) \cup (\text{int}(B) \cup \partial B) \subset \text{int}(A) \cup \text{int}(B).$$

Thus we obtain that $\bar{A} \cup \bar{B} = A \cup B = \text{int}(A) \cup \text{int}(B)$. By the connectedness of M we conclude that $M = \text{int}(A) \cup \text{int}(B)$. \square

Theorem 1.1 follows from the next result together with Theorem B and Lemma 3.1. Theorems 3.1, 3.3 and Lemma 3.1 imply together Theorem 1.4.

Theorem 3.3. Let Σ be a closed manifold with finite fundamental group. Let $f : \Sigma^{m-1} \rightarrow M^m$ be an immersion in a Riemannian manifold M . Let $g : N \rightarrow M$ be an immersion such $\exp^\perp : \mathcal{N}_g \rightarrow M$ is a covering map. Assume that the normal geodesics of M tangent to $f(\Sigma)$ do not intersect $g(N)$ orthogonally. Then M is complete, N is compact with finite fundamental group and the following conditions hold:

- (1) if $m - n = 1$ then Σ and N have the same universal covering, and the homomorphism $f_*^i : \pi_i(\Sigma) \rightarrow \pi_i(M)$ is a monomorphism for $i = 1$ and an isomorphism for $i \geq 2$;
- (2) if $m - n \geq 2$ then it holds that:
 - (a) $f(\Sigma) \cap g(N) = \emptyset$;
 - (b) M is noncompact with finite fundamental group;
 - (c) the homomorphism $\iota_*^i : \pi_i(f(\Sigma)) \rightarrow \pi_i(M)$, induced by the inclusion $\iota : f(\Sigma) \rightarrow M$, is surjective for $1 \leq i \leq m - n - 1$;
 - (d) if $m - n = 2$ then $f_*^i : \pi_i(\Sigma) \rightarrow \pi_i(M)$ is a monomorphism for $i = 2$ and an isomorphism for $i \geq 3$;
- (3) if $m - n \geq 3$ then the following statements hold:

- (a) the map $f_*^i : \pi_i(\Sigma) \rightarrow \pi_i(M)$ is a monomorphism for $i = 1$, an isomorphism for $2 \leq i \leq m - n - 2$ and an epimorphism for $i = m - n - 1$;
(b) if f is an embedding then $f_*^1 : \pi_1(\Sigma) \rightarrow \pi_1(M)$ is surjective and this, together with Item (a) above, implies that f is $(m - n - 1)$ -connected.

Proof. We first prove that M is complete, and N is compact with finite fundamental group.

$$(3) \quad \begin{array}{ccccc} \tilde{\Sigma} & \xrightarrow{\tilde{f}} & \tilde{M} = \mathcal{N}_{\tilde{g}} & \xleftarrow{\tilde{g}} & \tilde{N} \\ \downarrow \nu & & \downarrow \varphi & \nearrow \tilde{g} & \downarrow \mu \\ \Sigma & \xrightarrow{f} & M & \xleftarrow{g} & N \end{array}$$

$\begin{array}{c} \text{A curved arrow labeled } P \text{ points from } \tilde{M} \text{ to } M. \\ \text{A vertical arrow labeled } \exp_g^\perp \text{ points from } \mathcal{N}_g \text{ to } M. \end{array}$

In fact, let $\mu : \tilde{N} \rightarrow N$ be the universal covering and consider the map $\check{g} = g \circ \mu$. For $\tilde{z} \in \tilde{N}$ and $z = \mu(\tilde{z})$, set $W_z = (dg_z(T_z N))^\perp$ and $W_{\tilde{z}} = (d\check{g}_{\tilde{z}}(T_{\tilde{z}} \tilde{N}))^\perp$. Since $d\check{g}_{\tilde{z}} = dg_{\mu(\tilde{z})} \cdot d\mu_{\tilde{z}}$ and $d\mu_{\tilde{z}}$ is an isomorphism we obtain that $d\check{g}_{\tilde{z}}(T_{\tilde{z}} \tilde{N}) = dg_z(T_z N)$, hence $W_z = W_{\tilde{z}}$. Since $\mathcal{N}_{\tilde{g}} = \{(\tilde{z}, v) \mid \tilde{z} \in \tilde{N}, v \in W_{\tilde{z}}\}$ and $\mathcal{N}_g = \{(z, v) \mid z \in N, v \in W_z\}$, we conclude that $\varphi : \mathcal{N}_{\tilde{g}} \rightarrow \mathcal{N}_g$ given by $\varphi(\tilde{z}, v) = (\mu(\tilde{z}), v)$ is a covering map. The manifold $\tilde{M} = \mathcal{N}_{\tilde{g}}$ is simply-connected since it is strongly deformation retracted to $\tilde{N} \times \{0\}$. Thus $P = \exp_g^\perp \circ \varphi : \tilde{M} \rightarrow M$ is the universal covering of M . Consider on \tilde{M} and on \tilde{N} the induced metrics by P and μ , respectively.

Set $\tilde{g} : \tilde{N} \rightarrow \tilde{M}$ given by $\tilde{g}(\tilde{z}) = (\tilde{z}, 0)$. Notice that $(P \circ \tilde{g})(\tilde{z}) = \exp_g^\perp(\varphi(\tilde{z}, 0)) = \exp_g^\perp(z, 0) = g(z) = (g \circ \mu)(\tilde{z}) = \check{g}(\tilde{z})$, hence \tilde{g} is a lifting of \check{g} and g . Set $\tilde{\gamma} : [0, +\infty) \rightarrow \tilde{M}$ given by $\tilde{\gamma}(t) = (\tilde{z}, tv)$, where $v \in W_{\tilde{z}} = W_z$. Set $\gamma = P \circ \tilde{\gamma}$. Notice that $\gamma(t) = \exp_g^\perp(\mu(\tilde{z}), tv) = \exp_g^\perp(z, tv)$, hence γ is the geodesic on M with $\gamma(0) = g(z)$ and $\gamma'(0) = v$. Furthermore we have that $P(\tilde{N} \times \{0\}) = \exp_g^\perp(\mu(\tilde{N}) \times \{0\}) = g(N)$. Since $\gamma = P \circ \tilde{\gamma}$ is orthogonal to $g(N)$ and P is a local isometry, we have that $\tilde{\gamma}$ is a geodesic on \tilde{M} which is orthogonal to $\tilde{N} \times \{0\}$. Set $w = \tilde{\gamma}'(0)$. We have $v = \gamma'(0) = dP_{(\tilde{z}, 0)} \tilde{\gamma}'(0) = dP_{(\tilde{z}, 0)} w$. Then it holds that $\exp_g^\perp(\tilde{z}, w) = \tilde{\gamma}(1) = (\tilde{z}, v) = (\tilde{z}, dP_{(\tilde{z}, 0)} w)$, hence \exp_g^\perp is bijective. Since by hypothesis \exp^\perp is a covering map, we have that g is free of focal points, hence by induced metrics we obtain that \tilde{g} is also free of focal points. Thus the map $\exp_g^\perp : \mathcal{N}_{\tilde{g}} \rightarrow \tilde{M}$ is a local diffeomorphism. Since it is also bijective we conclude that \exp_g^\perp is a diffeomorphism.

Consider the universal covering $\nu : \tilde{\Sigma} \rightarrow \Sigma$ with induced metric. The map f admits a lifting $\tilde{f} : \tilde{\Sigma} \rightarrow \tilde{M}$. Since Σ is compact with finite fundamental group we have that $\tilde{\Sigma}$ is compact. Since we are using induced metrics we have that normal geodesics tangent to $\tilde{f}(\tilde{\Sigma})$ do not intersect $\tilde{g}(\tilde{N})$ orthogonally. Since $\exp_g^\perp : \mathcal{N}_{\tilde{g}} \rightarrow \tilde{M}$ is a diffeomorphism, Theorem 3.1 applies for $\tilde{f} : \tilde{\Sigma} \rightarrow \tilde{M}$ and $\tilde{g} : \tilde{N} \rightarrow \tilde{M}$. Thus we obtain that \tilde{M} is complete and \tilde{N} is compact. In particular Theorem 3.2 also applies for $\tilde{f} : \tilde{\Sigma} \rightarrow \tilde{M}$ and $\tilde{g} : \tilde{N} \rightarrow \tilde{M}$. Furthermore, since P and μ are locally isometric covering maps we conclude that M is complete and N is compact with finite fundamental group.

Now we will prove that f_*^1 is a monomorphism if $m - n \neq 2$. This will prove part of Items (1) and (3)-(a). Under this condition Theorem 3.2 says that \tilde{f} is an embedding. We take a continuous closed curve $\alpha : [0, 1] \rightarrow \Sigma$ such that $\beta = f \circ \alpha$ is trivial on $\pi_1(M)$. Let $\tilde{\alpha} : [0, 1] \rightarrow \tilde{\Sigma}$ be a lifting of α . Consider the curve $\tilde{\beta} = \tilde{f} \circ \tilde{\alpha} : [0, 1] \rightarrow \tilde{M}$. Notice that $P \circ \tilde{\beta} = (P \circ \tilde{f}) \circ \tilde{\alpha} = f \circ (\nu \circ \tilde{\alpha}) = f \circ \alpha = \beta$, hence $\tilde{\beta}$ is a lifting of β . Since β is trivial on $\pi_1(M)$ it follows that $\tilde{\beta} : [0, 1] \rightarrow \tilde{M}$ is a closed curve. From the equality $\tilde{\beta} = \tilde{f} \circ \tilde{\alpha}$ and the fact that \tilde{f} is injective we easily see that $\tilde{\alpha}$ is a closed curve, hence α is trivial on $\pi_1(\Sigma)$. This implies that f_*^1 is a monomorphism.

To complete the proof of Item (1) we assume that $m - n = 1$. By Theorem 3.2 we obtain that $\tilde{\Sigma}$ is diffeomorphic to \tilde{N} , and \tilde{f} is a homotopy equivalence. Thus Σ and N have the same universal covering and $\tilde{f}_*^i : \pi_i(\tilde{\Sigma}) \rightarrow \pi_i(\tilde{M})$ is an isomorphism for all i . Since P and ν are covering maps and $P \circ \tilde{f} = f \circ \nu$ we conclude that $f_*^i : \pi_i(\Sigma) \rightarrow \pi_i(M)$ is an isomorphism for all $i \geq 2$. Item (1) is proved.

Now we complete the proof of Item (3)-(a). Assume that $m - n \geq 3$. By Theorem 3.2 we have that \tilde{f} is $(m - n - 1)$ -connected. In particular it holds that $\tilde{f}_*^i : \pi_i(\tilde{\Sigma}) \rightarrow \pi_i(\tilde{M})$ is an isomorphism for $2 \leq i \leq m - n - 2$ and an epimorphism for $i = m - n - 1$. We use again the equality $P \circ \tilde{f} = f \circ \nu$ to conclude that $f_*^i : \pi_i(\Sigma) \rightarrow \pi_i(M)$ is an isomorphism for $2 \leq i \leq m - n - 2$ and an epimorphism for $i = m - n - 1$. Item (3)-(a) is proved.

Item (2)-(a) follows directly from Lemma 3.2. Item (2)-(d) follows from Item (2) of Theorem 3.2 together with the equality $P \circ \tilde{f} = f \circ \nu$. Item (3)-(b) will follow from Item (2)-(c) together with the assumption that f is an embedding. Thus to prove Theorem 3.3 it remains only to prove Items (2)-(b) and (2)-(c).

From now on we assume that $m - n \geq 2$. We will first prove Item (2)-(b). Consider any liftings $\hat{f} : \tilde{\Sigma} \rightarrow \tilde{M}$ and $\hat{g} : \tilde{\Sigma} \rightarrow \tilde{M}$ of f and g , respectively. Since we are using induced metrics on \tilde{M} and \tilde{N} we have that \hat{g} is an isometric immersion free of focal points. We already proved that \tilde{M} is complete and \tilde{N} is compact, hence we conclude by Remark 1.1 that $\exp_{\hat{g}}^\perp : \mathcal{N}_{\hat{g}} \rightarrow \tilde{M}$ is a diffeomorphism. Thus for any $x \in \tilde{M} - \hat{g}(\tilde{N})$ there exists a unique normal geodesic $\gamma_x = \gamma_{x, \hat{g}} : [0, \infty) \rightarrow \tilde{M}$ that starts orthogonally from $\hat{g}(\tilde{N})$ with $\gamma_x(d_x) = x$, where d_x is the distance between x and $\hat{g}(\tilde{N})$. Let $S_\epsilon = S_{\epsilon, \hat{g}} \subset \tilde{M}$ be the boundary of the ϵ -tubular neighborhood of $\hat{g}(\tilde{N})$. Since ν, P, μ are locally isometric covering maps we have that the normal geodesics tangent to $\hat{f}(\tilde{\Sigma})$ do not intersect $\hat{g}(\tilde{N})$ orthogonally. Again by Lemma 3.2 we have that $\hat{f}(\tilde{\Sigma}) \cap \hat{g}(\tilde{N}) = \emptyset$. By Lemma 2.1, the map $F : \tilde{\Sigma} \rightarrow S_\epsilon$ given by $F(p) = \gamma_{\hat{f}(p)}(\epsilon)$ is a local diffeomorphism. Since $\tilde{\Sigma}$ is compact and S_ϵ is connected we have that F is a covering map. Thus, for all $z \in S_\epsilon$, the geodesic γ_z ever intersects $\hat{f}(\tilde{\Sigma})$ and this occurs finitely many times. Let $t_{z, \hat{g}}$ be the time of the first contact and $T_{z, \hat{g}}$ the time of the last contact between γ_z and $\hat{f}(\tilde{\Sigma})$.

We claim that $T_{z, \hat{g}}$ and $t_{z, \hat{g}}$ depend continuously on $z \in S_\epsilon$. In fact, since F is a covering map, for any $z_0 \in S_\epsilon$, there exists a neighborhood $V \subset S_\epsilon$ of z_0 such that $F^{-1}(V)$ is a disjoint union of open sets U_1, \dots, U_k satisfying that, for all j , the restriction $F|_{U_j} : U_j \rightarrow V$ is a diffeomorphism, hence $f|_{U_j}$ is an embedding and, for any $z \in V$, the geodesic γ_z intersects transversely $f(U_j)$ at a unique point $\gamma_z(s_j(z))$. By transversality we know that $s_j : V \rightarrow (0, +\infty)$ is a smooth function. Thus the maps $z \in V \mapsto T_{z, \hat{g}}$ and $z \in V \mapsto t_{z, \hat{g}}$ are, respectively, maximum and minimum of smooth functions, hence we have that $T_{z, \hat{g}}$ and $t_{z, \hat{g}}$ depend continuously on z .

Define the sets

$$W_{\tilde{f},\tilde{g}} = \{\gamma_{z,\tilde{g}}(t) \mid z \in S_\epsilon \text{ and } 0 \leq t \leq T_{z,\tilde{g}}\},$$

and

$$w_{\tilde{f},\tilde{g}} = \{\gamma_{z,\tilde{g}}(t) \mid z \in S_\epsilon \text{ and } 0 \leq t \leq t_{z,\tilde{g}}\}.$$

Since $\exp_{\tilde{g}}^\perp$ is a diffeomorphism, and $T_{z,\tilde{g}}$ and $t_{z,\tilde{g}}$ depend continuously on z we easily see that $W_{\tilde{f},\tilde{g}}$ and $w_{\tilde{f},\tilde{g}}$ are tubular neighborhoods of $\hat{g}(\tilde{N})$ with C^0 -boundaries $\partial(W_{\tilde{f},\tilde{g}})$ and $\partial(w_{\tilde{f},\tilde{g}})$, respectively, which clearly are contained in $\hat{f}(\tilde{\Sigma})$. By definition of $W_{\tilde{f},\tilde{g}}$ it is easy to see that $\hat{f}(\tilde{\Sigma}) \subset W_{\tilde{f},\tilde{g}}$.

Fix liftings $\tilde{f} : \tilde{\Sigma} \rightarrow \tilde{M}$ and $\tilde{g} : \tilde{N} \rightarrow \tilde{M}$ of f and g , respectively. We claim that

$$(4) \quad \hat{g}(\tilde{N}) \subset W_{\tilde{f},\tilde{g}}, \text{ for any lifting } \hat{g} : \tilde{N} \rightarrow \tilde{M} \text{ of } g.$$

To prove this, assume by contradiction that $\hat{g}(\tilde{N}) \not\subset W_{\tilde{f},\tilde{g}}$, for some lifting \hat{g} . By Item (2) of Theorem 3.2 we have that $\tilde{f}(\tilde{\Sigma}) \cap \hat{g}(\tilde{N}) = \emptyset$, hence $\partial(W_{\tilde{f},\tilde{g}}) \cap \hat{g}(\tilde{N}) = \emptyset$. Thus the connectedness of $\hat{g}(\tilde{N})$ implies that $\hat{g}(\tilde{N}) \subset M - W_{\tilde{f},\tilde{g}}$. To get a contradiction we first will prove that $\partial(w_{\tilde{f},\tilde{g}}) \subset \partial(W_{\tilde{f},\tilde{g}})$. To see this we take a point $x \in \partial(w_{\tilde{f},\tilde{g}})$. Set $z = \gamma_{x,\tilde{g}}(\epsilon)$ and $\eta = \gamma_{x,\tilde{g}} = \gamma_{z,\tilde{g}}$. By definition of $w_{\tilde{f},\tilde{g}}$ we have that $x = \eta(t_{z,\tilde{g}})$. Notice that $\partial(w_{\tilde{f},\tilde{g}}) \subset \tilde{f}(\tilde{\Sigma}) \subset W_{\tilde{f},\tilde{g}}$, hence $x = \eta(t_{z,\tilde{g}}) \in W_{\tilde{f},\tilde{g}}$ and $\eta(0) \in \hat{g}(\tilde{N}) \subset \tilde{M} - W_{\tilde{f},\tilde{g}}$ and thus must exists $t_0 \in [0, t_{z,\tilde{g}}]$ such that $\eta(t_0) \in \partial(W_{\tilde{f},\tilde{g}})$. From the fact that $\partial(W_{\tilde{f},\tilde{g}}) \subset \tilde{f}(\tilde{\Sigma})$ and the definition of $t_{z,\tilde{g}}$ we have that $\eta([0, t_{z,\tilde{g}}]) \cap \partial W_{\tilde{f},\tilde{g}} \subset \eta([0, t_{z,\tilde{g}}]) \cap \tilde{f}(\tilde{\Sigma}) = \emptyset$, hence we obtain that $t_0 = t_{z,\tilde{g}}$ and $x \in \partial(W_{\tilde{f},\tilde{g}})$. Thus we proved that $\partial(w_{\tilde{f},\tilde{g}}) \subset \partial(W_{\tilde{f},\tilde{g}})$. Notice that $\partial(w_{\tilde{f},\tilde{g}})$ and $\partial(W_{\tilde{f},\tilde{g}})$ are connected closed C^0 -manifolds with the same dimension $m - 1$. Thus from the inclusion $\partial(w_{\tilde{f},\tilde{g}}) \subset \partial(W_{\tilde{f},\tilde{g}})$ we obtain that

$$(5) \quad \partial(w_{\tilde{f},\tilde{g}}) = \partial(W_{\tilde{f},\tilde{g}}).$$

We assert that

$$(6) \quad \text{int}(w_{\tilde{f},\tilde{g}}) \subset M - W_{\tilde{f},\tilde{g}}.$$

In fact, assume by contradiction that there exists $z \in \text{int}(w_{\tilde{f},\tilde{g}}) \cap W_{\tilde{f},\tilde{g}}$. We also have that $\hat{g}(\tilde{N}) \subset \text{int}(w_{\tilde{f},\tilde{g}}) \cap (M - W_{\tilde{f},\tilde{g}})$. Thus the connectedness of $\text{int}(w_{\tilde{f},\tilde{g}})$ implies that $\text{int}(w_{\tilde{f},\tilde{g}}) \cap \partial(W_{\tilde{f},\tilde{g}}) \neq \emptyset$, which contradicts (5). Thus, using (6), we obtain that

$$(7) \quad \text{int}(w_{\tilde{f},\tilde{g}}) \cap \text{int}(W_{\tilde{f},\tilde{g}}) = \emptyset.$$

Thus (5) and (7) imply together that $(w_{\tilde{f},\tilde{g}} \cup W_{\tilde{f},\tilde{g}})$ is an m -dimensional closed C^0 -manifold. We conclude that $w_{\tilde{f},\tilde{g}} \cup W_{\tilde{f},\tilde{g}} = \tilde{M}$, which contradicts the fact that \tilde{M} is noncompact. This proves (4).

Now we will prove Item (2)-(b). Fix $q \in N$ and $\tilde{q} \in \mu^{-1}(\{q\}) \subset \tilde{N}$. By using the Fundamental Lifting Theorem we obtain that

$$P^{-1}(\{g(q)\}) = \{\hat{g}(\tilde{q}) \mid \hat{g} : \tilde{N} \rightarrow \tilde{M} \text{ is a lifting of } g\} \subset W_{\tilde{f},\tilde{g}}.$$

From the compactness of $W_{\tilde{f},\tilde{g}}$ and the fact that $P^{-1}(\{g(q)\})$ is a discrete set we conclude that the set $P^{-1}(\{g(q)\})$ is finite, hence M has finite fundamental group and must be noncompact, since \tilde{M} is noncompact. Item (2)-(b) is proved.

Now we will prove Item (2)-(c). Fix liftings $\tilde{f} : \tilde{\Sigma} \rightarrow \tilde{M}$ and $\tilde{g} : \tilde{N} \rightarrow \tilde{M}$ of f and g , respectively. Fix $2 \leq i \leq m - n - 1$ and a point $z \in \partial(W_{\tilde{f}, \tilde{g}})$. Consider a continuous map $\psi : S^i \rightarrow M$ with $\psi(p) = P(z)$ for some point $p \in S^i$. We need to prove that there exists a homotopy ψ_t with $\psi_t(p) = P(z)$ for all $t \in [0, 1]$, $\psi_0 = \psi$ and $\psi_1(S^i) \subset f(\Sigma)$. Since $i \geq 2$ there exists a lifting $\tilde{\psi} : S^i \rightarrow \tilde{M}$ of ψ with $\tilde{\psi}(p) = z$. Since $\partial(W_{\tilde{f}, \tilde{g}})$ is the boundary of a tubular neighborhood of $\tilde{g}(\tilde{N})$ we have that the inclusion map $j : \partial(W_{\tilde{f}, \tilde{g}}) \rightarrow \tilde{M}$ is $(m - n - 1)$ -connected. Thus $\tilde{\psi}$ admits a homotopy $\tilde{\psi}_t$ with $\tilde{\psi}_t(p) = z$ for all $t \in [0, 1]$, $\tilde{\psi}_0 = \tilde{\psi}$ and $\tilde{\psi}_1(S^i) \subset \partial(W_{\tilde{f}, \tilde{g}})$. Thus $\psi_t = P \circ \tilde{\psi}_t$ is the desired homotopy, since $P(\partial(W_{\tilde{f}, \tilde{g}})) \subset P(\tilde{f}(\tilde{\Sigma})) = f(\Sigma)$.

Thus it remains to prove that the inclusion map $\iota : f(\Sigma) \rightarrow M$ satisfies that ι_*^1 is surjective. For this it suffices to prove that $P^{-1}(f(\Sigma))$ is connected. Consider the group $\text{Aut}(P)$ of the automorphisms $\alpha : \tilde{M} \rightarrow \tilde{M}$ of the covering map $P : \tilde{M} \rightarrow M$. Since we are using induced metrics we have that α is an isometry for any $\alpha \in \text{Aut}(P)$. By the Fundamental Lifting Theorem we have that

$$P^{-1}(f(\Sigma)) = \bigcup_{\alpha \in \text{Aut}(P)} (\alpha \circ \tilde{f})(\tilde{\Sigma}).$$

Thus to prove that $P^{-1}(f(\Sigma))$ is connected it suffices to show that

$$\tilde{f}(\tilde{\Sigma}) \cap (\alpha \circ \tilde{f})(\tilde{\Sigma}) \neq \emptyset, \text{ for any } \alpha \in \text{Aut}(P),$$

which will follow from the next assertion:

$$(8) \quad \partial(W_{\tilde{f}, \tilde{g}}) \cap \partial(W_{(\alpha \circ \tilde{f}), (\alpha \circ \tilde{g})}) \neq \emptyset, \text{ for any } \alpha \in \text{Aut}(P).$$

We assume by contradiction that assertion (8) is false. Then there exists $\alpha \in \text{Aut}(P)$ such that $\partial(W_{\tilde{f}, \tilde{g}}) \cap \partial(W_{\hat{f}, \hat{g}}) = \emptyset$ where $\hat{f} = \alpha \circ \tilde{f}$ and $\hat{g} = \alpha \circ \tilde{g}$. Since \hat{f} and \hat{g} are lifting maps of f and g , respectively, we have by (4) that $\tilde{g}(\tilde{N}) \subset W_{\tilde{f}, \tilde{g}} \cap W_{\hat{f}, \hat{g}}$. Since $W_{\tilde{f}, \tilde{g}}$ and $W_{\hat{f}, \hat{g}}$ are compact submanifolds with connected boundaries and \tilde{M} is noncompact it follows from Lemma 3.3 that either $W_{\tilde{f}, \tilde{g}} \subset W_{\hat{f}, \hat{g}}$ or $W_{\hat{f}, \hat{g}} \subset W_{\tilde{f}, \tilde{g}}$. Assume without loss of the generality that $W_{\tilde{f}, \tilde{g}} \subset W_{\hat{f}, \hat{g}}$. Thus the fact that $\partial(W_{\tilde{f}, \tilde{g}}) \cap \partial(W_{\hat{f}, \hat{g}}) = \emptyset$ implies that $W_{\tilde{f}, \tilde{g}} \subset \text{int}(W_{\hat{f}, \hat{g}})$, hence the volume $\text{vol}(W_{\tilde{f}, \tilde{g}}) < \text{vol}(W_{\hat{f}, \hat{g}})$.

Since α is an isometry it is easy to see that

$$(9) \quad \alpha(W_{\tilde{f}, \tilde{g}}) = W_{(\alpha \circ \tilde{f}), (\alpha \circ \tilde{g})} = W_{\hat{f}, \hat{g}}.$$

Thus we have that $\text{vol}(W_{\tilde{f}, \tilde{g}}) = \text{vol}(W_{\hat{f}, \hat{g}})$. This contradiction concludes the proof of Item (2)-(c). Theorem 3.3 is proved. \square

4. Examples

Let $p : M \rightarrow N$ be a vector bundle, where V_x denotes the fiber over x . It is well known that there exists a smooth map $x \in N \mapsto \langle \cdot, \cdot \rangle_x$, where $\langle \cdot, \cdot \rangle_x$ is an inner product on V_x . This map is usually called a fiber metric of p .

Proposition 4.1. *Let $p : M^m \rightarrow N^n$ be a vector bundle over the manifold N and let V_x denote the fiber over x . Fix a smooth fiber metric $x \in N \mapsto \langle \cdot, \cdot \rangle_x$, where $\langle \cdot, \cdot \rangle_x$ is an inner product on V_x . Let $S_x \subset V_x$ be the unit sphere centered at the origin and set $\mathcal{S}_1 = \bigcup_{x \in N} S_x$. Then there exists a Riemannian metric on the total space M such that the hypersurface \mathcal{S}_1 is totally geodesic and the null section*

$g : N \rightarrow N_0 = g(N) \subset M$ satisfies that $\exp^\perp : \mathcal{N}_g \rightarrow M$ is a diffeomorphism, hence g is free of focal points. If further N is compact then M is complete.

Proof. It is well known and easy to see that there exists a smooth distribution $z \in M \mapsto H_z \subset T_z M$ where H_z is an n -dimensional linear subspace which is transversal to the submanifold $V_{p(z)}$ at z and has the property that $H_{g(y)} = T_{g(y)} N_0$ for all $y \in N$. Fix $z \in M$ and set $p(z) = x$. We have a decomposition $T_z M = H_z \oplus T_z(V_x)$. Given $X \in T_z M$ we write $X = X_H + X_V$ with $X_H \in H_z$ and $X_V \in T_z(V_x)$. Given $X \in T_z(V_x)$, there exists a unique $K_z(X) \in V_x$ such that $X = \frac{d}{dt} \big|_{t=0} (z + tK_z(X))$. It is easy to see that $dp_z \big|_{H_z} : H_z \rightarrow T_x N$ and $K_z : T_z(V_x) \rightarrow V_x$ are isomorphisms. Fix any Riemannian metric ω_N on N . We define on M the Sasaki type Riemannian metric ω_1 (compare with [So]) by the equality

$$(10) \quad \omega_1(X, Y) = \omega_N(dp_z(X_H), dp_z(Y_H)) + \langle K_z(X_V), K_z(Y_V) \rangle_x,$$

for any $X, Y \in T_z M$.

Since $H_{g(x)} = T_{g(x)} N_0$ for all $x \in N$ we conclude that $N_0 = g(N)$ is orthogonal to the submanifold V_x . Thus we have that

$$\mathcal{N}_g = \mathcal{N}_{g, \omega_1} = \{(x, v) \mid x \in N, v \in (T_{g(x)} N_0)^\perp\} = \{(x, v) \mid x \in N, v \in T_{g(x)} V_x\}.$$

Thus we have a natural diffeomorphism $\varphi : \mathcal{N}_g \rightarrow M$ given by $\varphi(x, v) = K_{g(x)}(v)$. In fact, it is easy to see that the inverse map satisfies $\varphi^{-1}(z) = (p(z), (K_{g(p(z))})^{-1}(z))$. The map φ is in fact an isomorphism between vector bundles.

We claim that $\mathcal{S}_1 = \varphi(\mathcal{N}_g^1)$, where \mathcal{N}_g^1 is the unit normal bundle of g . In fact, by using (10), we obtain the following equivalences:

$$\begin{aligned} (x, v) \in \mathcal{N}_g^1 &\iff x \in N, v \in T_{g(x)} V_x \text{ and } \omega_1(v, v) = 1 \\ &\iff (x, v) \in \mathcal{N}_g \text{ and } \langle K_{g(x)}(v), K_{g(x)}(v) \rangle_x = 1 \\ &\iff (x, v) \in \mathcal{N}_g \text{ and } \langle \varphi(x, v), \varphi(x, v) \rangle_x = 1 \\ &\iff (x, v) \in \mathcal{N}_g \text{ and } \varphi(x, v) \in S_x \\ &\iff (x, v) \in \mathcal{N}_g \text{ and } \varphi(x, v) \in \mathcal{S}_1 \\ &\iff (x, v) \in \varphi^{-1}(\mathcal{S}_1). \end{aligned}$$

We conclude that $\mathcal{N}_g^1 = \varphi^{-1}(\mathcal{S}_1)$, hence $\mathcal{S}_1 = \varphi(\mathcal{N}_g^1)$. Our claim is proved.

In fact it can be proved that the normal exponential map $\exp_{g, \omega_1}^\perp : \mathcal{N}_g \rightarrow M$, associated with the map g and the metric ω_1 , coincides with the diffeomorphism φ . This can be done by making computations similar to those for the Sasaki metrics in TN and \mathcal{N}_g (see [Do], [GK], [BY]). To avoid these computations we will complete the proof of Proposition 4.1 without using this fact. We will just consider on M the metric ω_2 induced by the diffeomorphism φ from the Sasaki metric on \mathcal{N}_g (see [BY]). Consider the geodesic $\bar{\gamma} : [0, +\infty) \rightarrow \mathcal{N}_g$ given by $\bar{\gamma}(t) = (x, tv)$. It is well known that $\bar{\gamma}'(0)$ is orthogonal to $N \times \{0\}$ with respect to the Sasaki metric. This implies that the map $\gamma = \varphi \circ \bar{\gamma}$ is an ω_2 -geodesic which is ω_2 -orthogonal to $\varphi(N \times \{0\}) = K_{g(N)}(0) = g(N) = N_0$. Thus we have that $\exp_{g, \omega_2}^\perp(x, tv) = \gamma(t) = \varphi(x, tv)$, hence $\exp_{g, \omega_2}^\perp = \varphi$, hence \exp_{g, ω_2}^\perp is a diffeomorphism and even an isometric isomorphism between the vector bundles \mathcal{N}_g and M . For this metric it is easy to see that the normal bundle $\mathcal{N}_{g, \omega_2}$ coincides with \mathcal{N}_g and it holds that

$$(11) \quad \exp_{g, \omega_2}^\perp(x, tv) = \varphi(x, tv) = tK_{g(x)}(v).$$

We set:

$$\begin{aligned}\mathcal{S}_r &= \exp_{g, \omega_2}^\perp(\{(x, v) \in \mathcal{N}_g \mid \omega_1(v, v) = r^2\}), \\ \mathcal{B}_r &= \exp_{g, \omega_2}^\perp(\{(x, v) \in \mathcal{N}_g \mid \omega_1(v, v) < r^2\}).\end{aligned}$$

Since $\exp_{g, \omega_2}^\perp : \mathcal{N}_g \rightarrow M$ is a diffeomorphism, it is easy to see that a geodesic $\gamma : \mathbb{R} \rightarrow M$ satisfies

$$\begin{aligned}(12) \quad \gamma \text{ is } \omega_2\text{-orthogonal to } N_0 &\iff \gamma \text{ is } \omega_2\text{-orthogonal to } \mathcal{S}_r \text{ for some } r > 0 \\ &\iff \gamma \text{ is } \omega_2\text{-orthogonal to } \mathcal{S}_r \text{ for any } r > 0.\end{aligned}$$

With respect to the metric ω_2 , by using (12) it is easy to see that if σ_1, σ_2 are geodesics orthogonal to \mathcal{S}_1 then they may be defined on all \mathbb{R} (since they are reparameterizations of geodesics orthogonal to $g(N)$) and we also have that either

$$(13) \quad \sigma_1(\mathbb{R}) = \sigma_2(\mathbb{R}), \sigma_1(\mathbb{R}) \cap \sigma_2(\mathbb{R}) = \emptyset \text{ or } \sigma_1(\mathbb{R}) \cap \sigma_2(\mathbb{R}) = \{z\} \subset N_0.$$

Thus we conclude from (13) that $M - N_0$ is an open tubular neighborhood of \mathcal{S}_1 in the general sense of Definition 1.1. More precisely, if we consider the inclusion map $\iota : \mathcal{S}_1 \rightarrow M$ we have that

$$\exp_{\iota, \omega_2}^\perp|_W : W \rightarrow M - N_0$$

is a diffeomorphism, where $W = \{(z, t\nu(z)) \in \mathcal{N}_{\iota, \omega_2} \mid t \in (-1, +\infty)\}$ and $\nu(z) \in T_z M$ is the ω_2 -unitary vector orthogonal to \mathcal{S}_1 which points outwards the set \mathcal{B}_1 .

For $0 < s < 1$, let $U_s = U_{s, \omega_2}$ denote an open s -tubular neighborhood of \mathcal{S}_1 with respect to the metric ω_2 . From (12) and the facts that $\exp_{g, \omega_2}^\perp : \mathcal{N}_g \rightarrow M$ and $\exp_{\iota, \omega_2}^\perp|_W : W \rightarrow M - N_0$ are diffeomorphisms it is easy to see that

$$(14) \quad \partial U_s = \mathcal{S}_{1-s} \cup \mathcal{S}_{1+s} \text{ for all } 0 < s < 1.$$

Now we begin the construction of a metric on M such that \mathcal{S}_1 becomes totally geodesic. We consider on $\mathcal{N}_{\iota, \omega_2}$ the Sasaki metric and introduce on $M - N_0$ the metric ω_3 induced by the diffeomorphism $\exp_{\iota, \omega_2}^\perp|_W$. Notice that \mathcal{S}_1 is a totally geodesic hypersurface of $M - N_0$ if we consider the metric ω_3 . Furthermore we have trivially that, for any $0 < s < 1$ it holds that

$$(15) \quad U_s = U_{s, \omega_2} = U_{s, \omega_3}.$$

Fix $0 < \delta < \epsilon < 1$. We consider a smooth bump function $\zeta : M \rightarrow [0, 1]$ such that $\zeta(z) = 1$ on U_δ and $\zeta(z) = 0$ outside U_ϵ . We define on M the metric

$$(16) \quad \omega = (1 - \zeta)\omega_2 + \zeta\omega_3.$$

Thus the hypersurface \mathcal{S}_1 is totally geodesic relatively to the metric ω . Furthermore, take an ω_2 -geodesic γ which starts from N_0 orthogonally with respect to ω_2 . By (12) we see that γ is ω_2 -orthogonal to \mathcal{S}_1 , hence it is also ω_3 -orthogonal to \mathcal{S}_1 and it holds that $\omega_2(\gamma'(t), \gamma'(t)) = \omega_3(\gamma'(t), \gamma'(t))$, since the Sasaki metric ω_3 preserves orthogonality and length on radial directions. By (16) we conclude that

$$(17) \quad \omega(\gamma'(t), \gamma'(t)) = \omega_2(\gamma'(t), \gamma'(t)) = \omega_3(\gamma'(t), \gamma'(t)).$$

Our goal now is to prove that $\exp_{g, \omega}^\perp : \mathcal{N}_g \rightarrow M$ is a diffeomorphism (in particular $g : N \rightarrow (M, \omega)$ will be free of focal points). For this it suffices to show that $\varphi = \exp_{g, \omega}^\perp$. From the construction of the metric ω it follows that if $\omega_2(v, v) < 1 - \epsilon$ then it holds that $\varphi(x, v) = \exp_{g, \omega}^\perp(x, v)$. Thus we just need to show that for all $(x, v) \in \mathcal{N}_g - (N \times \{0\})$ it holds that the curve $\gamma = \gamma_{x, v} : [0, +\infty) \rightarrow V_x \subset M$ given

by $\gamma(t) = \varphi(x, tv)$ is an ω -geodesic (we already know that γ is an ω_2 -geodesic by (11)).

We fix an ω_2 -geodesic $\gamma = \gamma_{x,v}$ as above with $\omega_2(v, v) = 1$. We know that $\gamma|_{(0, +\infty)}$ is an ω_3 -geodesic. By (14), (15) and (16) we just need to prove that $\gamma|_{([1-\epsilon, 1-\delta] \cup [1+\delta, 1+\epsilon])}$ is an ω -geodesic. We will prove that $\gamma|_{[1+\delta, 1+\epsilon]}$ is an ω -geodesic, since the other case is similar. Fix $t \in [1+\delta, 1+\epsilon]$. By the Gauss Lemma we have that

$$(18) \quad \omega_2(\gamma'(t), \eta) = 0 \iff \eta \in T_{\gamma(t)}\mathcal{S}_t.$$

Set $t = 1 + s$ with $\delta < s < \epsilon$. From the Gauss Lemma, we obtain from (14) and (15) that

$$(19) \quad \begin{aligned} \omega_3(\gamma'(t), \eta) = 0 &\iff \eta \in T_{\gamma(t)}(\partial U_{s, \omega_3}) = T_{\gamma(t)}(\mathcal{S}_{1-s} \cup \mathcal{S}_{1+s}) \\ &= T_{\gamma(t)}(\mathcal{S}_{2-t} \cup \mathcal{S}_t) = T_{\gamma(t)}\mathcal{S}_t. \end{aligned}$$

Thus from (16), (18) and (19) it holds that if $\eta \in T_{\gamma(t)}\mathcal{S}_t$ then $\omega(\gamma'(t), \eta) = 0$. Thus we have the inclusion $T_{\gamma(t)}\mathcal{S}_t \subset \Omega_t = \{\eta \in T_{\gamma(t)}M \mid \omega(\gamma'(t), \eta) = 0\}$ between $(m-1)$ -dimensional linear spaces, which implies that $\Omega_t = T_{\gamma(t)}\mathcal{S}_t$, hence it holds that

$$(20) \quad \omega(\gamma'(t), \eta) = 0 \iff \omega_2(\gamma'(t), \eta) = 0 \iff \omega_3(\gamma'(t), \eta) = 0.$$

Now we are in condition to prove that γ is an ω -geodesic at $t \in [1+\delta, 1+\epsilon]$. On a small neighborhood of $\gamma(t)$ we consider the smooth ω -unitary vector field X given by the equation $X(\gamma_{y,u}(s)) = \gamma'_{y,u}(s)$ with $\omega_2(u, u) = 1$ and y in a neighborhood of $x \in N$. Note that (17) implies that

$$(21) \quad \omega(X, X) = \omega_2(X, X) = \omega_3(X, X) = 1.$$

Let ∇, ∇^2 and ∇^3 denote, respectively, the Levi-Civita connections associated to ω, ω_2 and ω_3 , respectively. Since $\omega(X, X) = 1$ we have that $\omega(\nabla_X X, X) = 0$. Thus we only need to prove that $\omega(\nabla_X X, \eta) = 0$ at $\gamma(t)$, for any $\eta \in \Omega_t$. Fix $\eta_0 \in \Omega_t$ and extend it to a smooth vector field η on a neighborhood of $\gamma(t)$. From (21) we have that

$$(22) \quad \eta(\omega(X, X)) = \eta(\omega_2(X, X)) = \eta(\omega_3(X, X)) = 0.$$

Thus by using the Koszul formula and the fact that $\gamma|_{[1+\delta, 1+\epsilon]}$ is a geodesic with respect to ω_2 and ω_3 we have that

$$(23) \quad 0 = \omega_2(\nabla_X^2 X, \eta) = X(\omega_2(X, \eta)) - \omega_2([X, \eta], X),$$

$$(24) \quad 0 = \omega_3(\nabla_X^3 X, \eta) = X(\omega_3(X, \eta)) - \omega_3([X, \eta], X).$$

By using the Koszul Formula, it follows from (16), (20), (22), (23), (24) that:

$$(25) \quad \begin{aligned} \omega(\nabla_X X, \eta) &= X(\omega(X, \eta)) - \omega([X, \eta], X) \\ &= X\{(1-\zeta)\omega_2(X, \eta) + \zeta\omega_3(X, \eta)\} \\ &\quad - (1-\zeta)\omega_2([X, \eta], X) - \zeta\omega_3([X, \eta], X) \\ &= X(\zeta)\{\omega_3(X, \eta) - \omega_2(X, \eta)\} \\ &\quad + (1-\zeta)\{X(\omega_2(X, \eta)) - \omega_2([X, \eta], X)\} \\ &\quad + \zeta\{X(\omega_3(X, \eta)) - \omega_3([X, \eta], X)\} \\ &= X(\zeta)\{\omega_3(X, \eta) - \omega_2(X, \eta)\} \\ &= 0. \end{aligned}$$

We conclude that γ is an ω -geodesic, hence $\varphi = \exp_{g,\omega}^\perp$ and therefore the map $\exp_{g,\omega}^\perp : \mathcal{N}_g \rightarrow M$ is a diffeomorphism.

Finally, if we have the additional hypothesis that N is compact, we obtain from Corollary 3.2 that M is complete. Proposition 4.1 is proved. \square

Corollary 4.1. *Let M, N be Riemannian manifolds and $g : N \rightarrow M$ an immersion such that $\exp_g^\perp : \mathcal{N}_g \rightarrow M$ is a diffeomorphism. Then there exists a Riemannian metric ω on M such that $\exp_{g,\omega}^\perp : \mathcal{N}_g \rightarrow M$ is a diffeomorphism and $\mathcal{S}_1 = \exp^\perp(\mathcal{N}_g^1) \subset M$ is totally geodesic.*

Proof. Let $\iota : N \rightarrow \mathcal{N}_g$ be the embedding given by $\iota(x) = (x, 0)$. Apply Proposition 4.1 to \mathcal{N}_g and obtain a metric ω_1 on \mathcal{N}_g such that $\exp_{\iota,\omega_1}^\perp : \mathcal{N}_\iota \rightarrow \mathcal{N}_g$ is a diffeomorphism and \mathcal{N}_g^1 becomes totally geodesic. Now we just introduce the metric on M induced by the diffeomorphism $\exp_g^\perp : \mathcal{N}_g \rightarrow M$. Corollary 4.1 is proved. \square

Proposition 4.2. *Take $B = (\mathbb{R}^k, ds^2)$ and $ds^2 = dr^2 + \sigma^2(r)d\theta^2$ being the metric introduced in Example 1.4. Let $\mathcal{S} = S^{k-1} \subset B$ be the totally geodesic unit sphere and N' any manifold. Consider a warped product $M = B \times_\rho N'$ and assume that the gradient $(\nabla \rho)|_{\mathcal{S}}$ is tangent to \mathcal{S} . Then the inclusion maps: $f : \Sigma = \mathcal{S} \times N' \rightarrow M$ and $g : N = \{0\} \times N' \rightarrow M$ have the properties that f is totally geodesic and that the normal exponential map $\exp^\perp : \mathcal{N}_g \rightarrow M$ is a diffeomorphism, hence g is free of focal points.*

Proof. We recall some notations and facts about warped products. For a positive smooth function $\rho : B \rightarrow (0, \infty)$, the warped product $M = B \times_\rho N'$ is the product manifold $B \times N'$ with the metric

$$(26) \quad \langle Z, W \rangle = ds^2(d\pi_B(Z), d\pi_B(W)) + (\rho \circ \pi_B)^2 \langle d\pi_{N'}(Z), d\pi_{N'}(W) \rangle_{N'},$$

where $\pi_B, \pi_{N'}$ are the canonical projections of $B \times N'$ onto its corresponding factors and $\langle \cdot, \cdot \rangle_{N'}$ is the metric of N' .

Consider the decomposition $TM = \mathcal{D}(B) \oplus \mathcal{D}(N')$, where $\mathcal{D}(B)$ and $\mathcal{D}(N')$ are the subbundles of TM given by the distributions corresponding to the product foliations determined by B and N' , respectively. For tangent vector fields X on B and Y on N' there exist unique lifts $X^h \in \mathcal{D}(B)$ and $Y^v \in \mathcal{D}(N')$ satisfying that $d\pi_B(X^h) = X$ and $d\pi_{N'}(Y^v) = Y$. For tangent vector fields X on B and Y on N' , the Levi-Civita connection $\bar{\nabla}$ of the warped product $M = B \times_\rho N'$ is related to the Levi-Civita connections ∇^B and $\nabla^{N'}$ of its corresponding factors by the following equations (see [O'N]):

- (i) $\bar{\nabla}_{X^h} X^h = (\nabla_X^B X)^h$;
- (ii) $\bar{\nabla}_{X^h} Y^v = \bar{\nabla}_{Y^v} X^h = ds^2(X, \eta) Y^v$;
- (iii) $\bar{\nabla}_{Y^v} Y^v = -\langle Y, Y \rangle_{N'} \eta^h + (\nabla_Y^{N'} Y)^v$;

where $\eta = \nabla(\log \rho)$ and ∇ is the gradient on B .

By (26) the horizontal fibres $B_x = B \times \{x\} \subset M$ with the induced metric are isometric to B . By (i) above the fibers B_x are totally geodesic submanifolds of M . We know that the exponential map $\exp : T_0 B \rightarrow B$ is a diffeomorphism, hence $\exp : T_{(0,x)} B_x \rightarrow B_x$ is also a diffeomorphism for all x . Furthermore by (26) we have that B_x is orthogonal to $N = \{0\} \times N'$, hence $\exp^\perp : \mathcal{N}_g \rightarrow M$ is a diffeomorphism.

We claim that the inclusion map $f : \Sigma = \mathcal{S} \times N' \rightarrow M$ is totally geodesic. In fact let $r : B \rightarrow [0, +\infty)$ be the distance function from the origin. Set $\mu = (\nabla r)^h$. We know that $\langle \mu, \mu \rangle = ds^2(\nabla r, \nabla r) = 1$. Since ∇r is B -orthogonal to \mathcal{S} we have that μ is orthogonal to $\mathcal{S} \times \{x\}$ for any $x \in N'$. We also have that $\mu = (\nabla r)^h$ is orthogonal to $\{p\} \times N'$ for all $p \in B$, hence we obtained a unitary vector field μ on $M - (\{0\} \times N')$ which is orthogonal to Σ . Now fix $(p, x) \in \Sigma$ and $Z \in T_{(p,x)}\Sigma$. We claim that there exists a local tangent vector field \bar{Z} on Σ extending Z such that $\bar{Z} = X^h + Y^v$ for some tangent vector fields X on B and Y on N' . In fact, since $\Sigma = \mathcal{S} \times N'$ we have a unique orthogonal decomposition $Z = Z_h + Z_v$ with $Z_h \in T_{(p,x)}(\mathcal{S} \times \{x\})$ and $Z_v \in T_{(p,x)}(\{p\} \times N')$. Extend $d\pi_B(Z_h)$ to a tangent vector field X on B with the property that $X|_{\mathcal{S}}$ is tangent to \mathcal{S} and extend $d\pi_{N'}(Z_v)$ to a tangent vector field Y on N' . Thus we have that $\bar{Z} = X^h + Y^v$ is tangent to Σ and $\bar{Z}(p, x) = Z$. By using (i), (ii) and (iii) above we obtain that

$$\begin{aligned} \langle \bar{\nabla}_{\bar{Z}} \bar{Z}, \mu \rangle_{(p,x)} &= \langle \bar{\nabla}_{X^h} X^h, (\nabla r)^h \rangle + 2 \langle \bar{\nabla}_{X^h} Y^v, (\nabla r)^h \rangle + \langle \bar{\nabla}_{Y^v} Y^v, (\nabla r)^h \rangle \\ &= ds^2(\nabla_X^B X, \nabla r)_p + 2 \langle \langle X, \eta \rangle Y^v, (\nabla r)^h \rangle - \langle \langle Y, Y \rangle_{N'} \eta^h, (\nabla r)^h \rangle \\ &= ds^2(\nabla_X^B X, \nabla r)_p - \langle Y, Y \rangle_{N'} ds^2(\eta, \nabla r)_p. \end{aligned}$$

We have that p belongs to the ds^2 -totally geodesic submanifold $\mathcal{S} \subset B$. By construction we have that $X|_{\mathcal{S}}$ is tangent to \mathcal{S} , hence it holds that $ds^2(\nabla_X^B X, \nabla r)_p = 0$. Since $\eta|_{\mathcal{S}}$ is by hypothesis tangent to \mathcal{S} and ∇r is ds^2 -orthogonal to \mathcal{S} we have that $ds^2(\eta, \nabla r)_p = 0$. Thus we conclude that $\langle \bar{\nabla}_{\bar{Z}} \bar{Z}, \mu \rangle_{(p,x)} = 0$, hence f is totally geodesic. \square

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